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Aggregation and Risk Measurement for a Portfolio of Exchangeable Bernoulli Risks, assuming Dependence Uncertainty

Étienne Marceau

École d'actuariat
Université Laval, Québec, Canada

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Faculté des sciences et de génie
École d'actuariat

Introduction

Portfolio and tasks

Portfolio of n risks $X_1,\ldots,X_n$ for an insurance company or a bank

- rv X_i : losses for the risk i, $i = 1,2,\ldots,n$.
- rv $S_n = X_1 + \ldots + X_n$: aggregate losses for the portfolio
- risk measure φ_κ: quantifying the global risk for the portfolio

Basic risk measures φ_κ $(\kappa \in]0,1[)$:

- $VaR_\kappa(S_n) = F_{S_n}^{-1}(\kappa) = \inf\{j \in \mathcal{A}_n, F_{S_n}(j) \geq \kappa\}$
- $TVaR_\kappa(S_n) = \frac{1}{1-\kappa}\int_\kappa^1 VaR_u(S_n)\,\mathrm{d}u$

General tasks for computing φ_κ:

- Step 1: Find the marginal distributions for X_1, ..., X_n
- Step 2: Find the joint distribution for $\underline{X} = (X_1,\ldots,X_n)$
- Step 3: Identify (if possible) and compute F_{S_n}
- Step 4: Compute a risk measure $\varphi_\kappa(S_n)$ associated to S_n

Introduction

Realistic assumptions for this talk

A1: Exchangeable Bernoulli rvs: Homogeneous portfolio

- individual losses: $X_i = bI_i$, with $I_i \sim Bern\,(q)$, $i = 1,2,\ldots,n$
- fixed amount of individual loss: $b > 0$
- realistic dependence structure: $\underline{I} = (I_1,\ldots,I_n) =$ vector of exchangeable Bernoulli rvs
- number of losses: $N_n = I_1 + \ldots + I_n =$ sum of exchangeable Bernoulli rvs
- aggregate losses: $S_n = b \times N_n$, with $N_n \in \mathcal{A}_n = \{0,1,\ldots,n\}$

A2: Dependence uncertainty: Partial information on the joint distribution of $\underline{I}$

- known expectation: $E[I_i] = q \in (0,1)$, for $i \in \{1,\ldots n\}$
- Pearson's correlation coefficient: $\rho_P(I_i,I_j) = \rho$, for $i \neq j \in \{1,2,\ldots,n\}$

Introduction

Main objectives

Main objectives :

- **Obj1**: Provide a brief overview of possible multivariate distributions for $\underline{I}$ and examine their impact on $TVaR_\kappa(S_n)$, for $\kappa \in (0,1)$
- **Obj2**: Find lower and upper (moment) bounds on $TVaR_\kappa(S_n)$ based on partial information on the joint distribution of $\underline{I}$

Applications in Actuarial Science and QF: Homogeneous portfolio of (credit) risks

- I_i: occurrence rv for risk i, $i = 1,2,\ldots,n$
- $I_i = 1$, if a loss (default) occurs; $I_i = 0$, otherwise
- N_n = number of losses (defaults)

Other applications: Biostatistics, Machine Learning, Etc.

Introduction

Numerical application of main objectives

Known values from the industry (McNeil et al.,2015)

- $E[I_i] = q = 0049$, for $i \in \{1,\ldots n\}$
- $\rho_P(I_i,I_j) = 0.0156$, for $i \neq j \in \{1,2,\ldots,n\}$

Find the answers to the following questions:

- **Obj1**: What are the possible multivariate distribution for $\underline{I}$ and their impact on $TVaR_\kappa(S_n)$, for $\kappa \in (0,1)$?
- **Obj2**: What are the values $\varphi_{min}(\kappa)$ and $\varphi_{max}(\kappa)$ such that

$$\varphi_{min}(\kappa) \leq TVaR_\kappa(S_n) \leq \varphi_{max}(\kappa) \tag{1}$$

for the given known values of q and ρ

Multivariate distributions for exchangeable Bernoulli rvs

Representation result by Bowman-George-Madsen

Define:

- Vector of probabilities: $\underline{\zeta} = (\zeta_0, \zeta_1, \ldots, \zeta_n)$ where
- $\zeta_k = \Pr(I_1 = 1, \ldots, I_k = 1)$, $k = 1, 2, \ldots, n$, with
- $\zeta_0 = 1 \geq \zeta_1 \geq \ldots \geq \zeta_n \geq 0$ and $q = \Pr(I_j = 1) = \zeta_1$, $j = 1, 2, \ldots, n$

Result: $\underline{\zeta}$ completely specifies the distributions of $\underline{I}$ and N_n

- pmf of $\underline{I}$:

$$\Pr(I_1 = i_1, \ldots, I_n = i_n) = \sum_{l=0}^{n-k} (-1)^l \binom{n-k}{l} \zeta_{k+l}$$

  with $i_l \in \{0, 1\}$, $l = 1, 2, \ldots, n$, be such that $\sum_{l=1}^{n} i_l = k \in \mathcal{A}_n$

- pmf of $N_n = I_1 + \ldots + I_n$:

$$\Pr(N_n = k) = \binom{n}{k} \sum_{l=0}^{n-k} (-1)^l \binom{n-k}{l} \zeta_{k+l}, \quad \text{for } k \in \mathcal{A}_n$$

Multivariate distributions for exchangeable Bernoulli rvs

Class of Exchangeable Bernoulli rvs with fixed mean

Important: Joint distribution of $\underline{I} \Longleftrightarrow$ univariate distribution of N_n

Fix: $\zeta_1 \in (0,1)$ and $n \geq 2$

Define: $\mathcal{C}_n(\zeta_1) =$ class of all joint cdf $F_{\underline{I}}$ of $\underline{I} = (I_1, \ldots, I_n)$ with

$$\Pr(I_i = 1) = \zeta_1,\ i = 1, 2, \ldots, n$$

- Pearson's correlation coeffcient: $\rho_P(I_{i_1}, I_{i_2}) = \frac{\zeta_2 - \zeta_1^2}{\zeta_1 - \zeta_1^2} \in \left[-\frac{1}{n-1}, 1\right]$
- Remark: pairs of rvs (I_{i_1}, I_{i_2}) can be negatively correlated
- Subclass of $\mathcal{C}_n(\zeta_1)$: $\mathcal{C}_n^+(\zeta_1) =$ class of $F_{\underline{I}} \in$ of $\underline{I} \in \mathcal{C}_n(\zeta_1)$ such that

$$\rho_P(I_{i_1}, I_{i_2}) \in [0, 1]$$

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Multivariate distributions for exchangeable Bernoulli rvs

List of approaches

List of approaches to construct multivariate distributions for $\underline{I}$:

- Approach based on a sufficient condition on $\underline{\zeta} = (\zeta_0, \zeta_1, \ldots, \zeta_n)$
- Approach based on DeFinetti's result
- Approach based on an exchangeable copula
- Approach based on latent exchangeable rvs
- Etc.

Treat yourself in e.g. Joe (1997, ch7), Gordy (2000), Frey and McNeil (2003), Embrechts et al. (2005), Bowman and George (1995, 2016), Kuk (2004), Mai & Shrerer (2014), etc.

Multivariate distributions for exchangeable Bernoulli rvs

Special cases

Special case #1: $\zeta_k = q^k \; (k = 0,1,\ldots,n)$

- $\Rightarrow I_1, \ldots, I_n$ are independent and $N_n \sim Binomial\,(n,q)$ and
- Notation: $F_{\underline{I}^{\perp}} \in \mathcal{C}_n\,(\zeta_1)$

Special case #2: $\zeta_k = \begin{cases} 1 & k = 0 \\ q & k = 1,2,\ldots,n \end{cases}$

- $\Rightarrow I_1, \ldots, I_n$ are comonotonic and $\frac{N_n}{n} \sim Bernoulli\,(q)$
- Notation: $F_{\underline{I}^{+}} \in \mathcal{C}_n\,(\zeta_1)$

Special case #3 (Fréchet Copula): $\zeta_k = \begin{cases} 1 & , k = 0 \\ \alpha \times q + (1-\alpha) \times q^k & , k = 1,2,\ldots,n \end{cases}, \alpha \in [0,1]$

- $\Rightarrow F_{\underline{I}} = \alpha F_{\underline{I}^{+}} + (1-\alpha)\,F_{\underline{I}^{\perp}}$ and $F_N = \alpha F_{N^{+}} + (1-\alpha)\,F_{N^{\perp}}$
- Mixture of special cases #1 and #2

Multivariate distributions for exchangeable Bernoulli rvs

Approach based of a sufficient condition

Objective: Define $F_{\underline{I}}$ and F_N using $\underline{\zeta}$

Sufficient condition (Masden (1993)):

- Let $\psi(t)$ be a completely comonotone function for $t \geq 0$.
- If $\zeta_k = \psi(k)$, then $f_{N_n}(k) \geq 0$, for $k \in \mathcal{A}_n$

Feller (1971): ψ is completely comonotone iff ψ is the LST of a strictly positive rv Y, i.e.,

$$\psi(t) = \mathcal{L}_Y(t) = E\left[\mathrm{e}^{-Yt}\right], \quad t \geq 0$$

Define $\zeta_k = \mathcal{L}_Y(rk)$ where r is fixed such that $\zeta_1 = \mathcal{L}_Y(r)$

- $\Rightarrow \zeta_k = \mathcal{L}_Y\left(k \times \mathcal{L}_Y^{-1}(\zeta_1)\right)$ where $r = \mathcal{L}_Y^{-1}(\zeta_1) = \mathcal{L}_Y^{-1}(q)$
- $\Rightarrow 0 \leq \rho_P(I_{i_1}, I_{i_2}) \leq 1$ and $F_{\underline{I}} \in \mathcal{C}_n^{+}(\zeta_1)$

Multivariate distributions for exchangeable Bernoulli rvs

Approach based of a sufficient condition

Example #1: $Y \sim Gamma\left(\frac{1}{\alpha},1\right)$ with $\mathcal{L}_Y(t) = \left(\frac{1}{1+t}\right)^{\frac{1}{\alpha}}$ and $\mathcal{L}_Y^{-1}(u) = u^{-\alpha} - 1$

- $\zeta_k = (k \times q^{-\alpha} - k + 1)^{-\frac{1}{\alpha}}$, for $k \in \mathcal{A}_n$
- $\alpha \in \mathbb{R}^+ =$ dependence parameter and $\alpha \to 0 \Rightarrow$ independence
- fixed $\rho_P(I_i,I_j) = \rho \Rightarrow$ only one value for α

Example #2: $Y \sim Stable\left(\frac{1}{\alpha},1\right)$ with $\mathcal{L}_Y(t) = \mathrm{e}^{-t^{\frac{1}{\alpha}}}$ and $\mathcal{L}_Y^{-1}(u) = (-\ln(u))^{\alpha}$

- $\zeta_k = \mathrm{e}^{-\ln(q)\left(k^{\frac{1}{\alpha}}\right)}$, for $k \in \mathcal{A}_n$
- $\alpha \in [1, \infty) =$ dependence parameter and $\alpha = 1 \Rightarrow$ independence
- fixed $\rho_P(I_i,I_j) = \rho \Rightarrow$ only one value for α

Multivariate distributions for exchangeable Bernoulli rvs

Approach based on DeFinetti's representation theorem

DeFinetti's representation theorem:

$$\Pr(I_1 = i_1, \ldots, I_n = i_n) = \int_0^1 \prod_{j=1}^{n} \theta^{i_j} (1-\theta)^{1-i_j} \, \mathrm{d}F_\Theta(\theta),$$

for $(i_1, \ldots, i_n) \in \{0,1\}^n$
Mixing rv Θ defined on $[0,1]$, with cdf F_Θ:

- $\Theta =$ unobservable rv (random environment)
- Given $\Theta = \theta$, $(I_1|\Theta = \theta) \sim Bern(\theta)$, …, $(I_n|\Theta = \theta) \sim Bern(\theta)$ are conditionally independent
- Given $\Theta = \theta$, $(N_n|\Theta = \theta) \sim Binom(n, \theta)$

Multivariate distributions for exchangeable Bernoulli rvs

Approach based on DeFinetti's representation theorem

Link between ζ_k and $E\left[\Theta^k\right]$: $\zeta_k = \Pr\left(I_1 = 1,\ldots,I_k = 1\right) = \int_0^1 \theta^k \mathrm{d}F_\Theta\left(\theta\right) = E\left[\Theta^k\right]$

Distribution of $N_n =$ mixed-binomial distribution

Remark on DeFinetti's representation theorem:

- only valid for $0 \leq \rho_P\left(I_{i_1},I_{i_2}\right) \leq 1 \Rightarrow F_{\underline{I}} \in \mathcal{C}_n^+\left(\zeta_1\right)$

Link with the approach based on $\mathcal{L}_Y$: define $\Theta = \mathrm{e}^{-Y}$, Y is a strictly positive rv

Multivariate distributions for exchangeable Bernoulli rvs

Approach based on DeFinetti's representation theorem

Well known-example proposed by Skellam (1948): $\Theta \sim Beta(a,b)$

- Distribution of N_n = beta-binomial distribution
- Frequently used in ActSci, QRM, biostatistic, etc.
- Fix $q = \zeta_1$ and $\rho_P(I_i, I_j) = \rho \Rightarrow$ values of the parameters (a, b)

Problems (see e.g. Kuk (2004)) with beta-binomial distribution:

- not sufficiently flexible
- non-robustness of its estimates to a misspecification of the correlation structure

Possible extension to account for negative correlation proposed by Prentice (1986)

Multivariate distributions for exchangeable Bernoulli rvs

Approach based on an exchangeable Archimedean copula

Define $F_{\underline{I}}$ by

$$F_{\underline{I}}\left(i_1,\ldots,i_n\right) = C\left(F_{I_1}\left(i_1\right),\ldots,F_{I_n}\left(i_n\right)\right)$$

where

- C is exchangeable Archimedean copula defined by

$$C\left(u_1,\ldots,u_n\right) = \mathcal{L}_Y\left(\mathcal{L}_Y^{-1}\left(u_1\right) + \ldots. + \mathcal{L}_Y^{-1}\left(u_n\right)\right)$$

- $\mathcal{L}_Y = E\left[\mathrm{e}^{-tY}\right]\left(t \geq 0\right) =$ LST of a strictly positive rv Y

Common mixture representation

$$F_{\underline{I}}\left(i_1,\ldots,i_n\right) = \int_0^\infty \mathrm{e}^{-y\mathcal{L}_Y^{-1}\left(F_{I_n}\left(i_n\right)\right)}\ldots\mathrm{e}^{-y\mathcal{L}_Y^{-1}\left(F_{I_n}\left(i_n\right)\right)}\mathrm{d}F_Y\left(y\right)$$

Multivariate distributions for exchangeable Bernoulli rvs

Approach based on an exchangeable Archimedean copula

Link with approach based on DeFinetti's representation:

- Define $\Theta = 1 - \mathrm{e}^{-Y \times \mathcal{L}_Y^{-1}(1-q)}$
- $\Pr(I_1 = i_1, \dots, I_n = i_n) = \int_0^1 \prod_{j=1}^{n} \theta^{i_j} (1-\theta)^{1-i_j} \, \mathrm{d}F_\Theta(\theta)$
- $\zeta_k = E\left[\Theta^k\right] = \sum_{j=0}^{k} \binom{k}{j} (-1)^j \mathcal{L}_Y\left(j \times \mathcal{L}_Y^{-1}(1-q)\right)$, $k \in \mathcal{A}_n$

Comparaison with the approach based on $\mathcal{L}_Y$:

- $\zeta_k = \mathcal{L}_Y\left(k \times \mathcal{L}_Y^{-1}(q_1)\right)$, $k \in \mathcal{A}_n$

Comment of the present approach:

- $0 \le \rho_P(I_{i_1}, I_{i_2}) \le 1 \Rightarrow F_{\underline{I}} \in \mathcal{C}_n^+(\zeta_1)$

Multivariate distributions for exchangeable Bernoulli rvs

Approach based on an exchangeable Archimedean copula

We can also define $\overline{F}_{\underline{I}}$ by

$$\overline{F}_{\underline{I}}\left(i_1,\ldots,i_n\right) = C\left(\overline{F}_{I_1}\left(i_1\right),\ldots,\overline{F}_{I_n}\left(i_n\right)\right)$$

where

- $C =$ exchangeable Archimedean copula defined by

$$C\left(u_1,\ldots,u_n\right) = \mathcal{L}_Y\left(\mathcal{L}_Y^{-1}\left(u_1\right) + \ldots. + \mathcal{L}_Y^{-1}\left(u_n\right)\right)$$

- $\mathcal{L}_Y = E\left[\mathrm{e}^{-tY}\right]$ $(t \geq 0)$ is the LST of a strictly positive rv Y

Comment: equivalent to the approach by sufficient condition

Dependence uncertainty for exchangeable Bernoulli rvs

Motivation

We need to be careful in the choice of the joint distribution of $\underline{I}$

Dependence uncertainty ⇒ possibility of model risk :

- Wrong choice about the dependence structure for $\underline{I}$
- ⇒ Wrong choice about the distribution of $N_n = I_1 + \ldots + I_n$
- ⇒ Strong impact on the values of risk measure $\varphi_\kappa(S_n)$, especially for $\kappa \in (0.9, 1)$
- ⇒ Strong impact on the risk assessment of the portfolio of n risks

Dependence uncertainty for exchangeable Bernoulli rvs

Problem's setting

Fix: $\zeta_1 \in (0,1)$ and $\zeta_2 \in \left(\zeta_{2,n}^{\min}, \zeta_1\right)$ where $\zeta_{2,n}^{\min} = \zeta_1^2 - \frac{(\zeta_1 - \zeta_1^2)^2}{n-1}$

Define: $C_n(\zeta_1, \zeta_2) \subset C_n(\zeta_1) =$ class of all multivariate cdf $F_{\underline{I}}$ of $\underline{I}$ such that

- $\Pr(I_i = 1) = \zeta_1$, for $i = 1, 2, \ldots, n$
- $\Pr(I_{i_1} = 1, I_{i_2} = 1) = \zeta_2$, for $i_1, i_2 = 1, 2, \ldots, n$ and $i_1 \neq i_2$
- condition of existence: $\zeta_2 \in \left(\zeta_{2,n}^{\min}, \zeta_1\right)$

Subclass of $\mathcal{C}_n(\zeta_1, \zeta_2)$: $\mathcal{C}_n^+(\zeta_1, \zeta_2) =$ class of $F_{\underline{I}} \in$ of $\underline{I} \in \mathcal{C}_n(\zeta_1, \zeta_2)$ such that

$$\zeta_2 \in (\zeta_1^2, \zeta_1)$$

Dependence uncertainty for exchangeable Bernoulli rvs

Problem's setting

Problem: for a fixed $\kappa \in (0,1)$, find (compute) $\underline{TVaR}_{\kappa}^{(\zeta_1,\zeta_2)}$ and $\overline{TVaR}_{\kappa}^{(\zeta_1,\zeta_2)}$ such that

$$\underline{TVaR}_{\kappa}^{(\zeta_1,\zeta_2)} \leq TVaR_{\kappa}(S_n) \leq \overline{TVaR}_{\kappa}^{(\zeta_1,\zeta_2)}$$

for any $\underline{I}$ with $F_{\underline{I}} \in \mathcal{C}_n\left(\zeta_1,\zeta_2\right)$

However: $\nexists$ vector of exchangeable Bernoulli rvs $\underline{I}^*$ with $F_{\underline{I}^*} \in \mathcal{C}_n\left(\zeta_1,\zeta_2\right)$ such that

$$\overline{TVaR}_{\kappa}^{(\zeta_1,\zeta_2)} = b \times TVaR_{\kappa}\left(I_1^* + \ldots + I_n^*\right) \text{ for all } \kappa \in [0,1]$$

However: $\nexists$ vector of exchangeable Bernoulli rvs $\underline{I}^{**}$ with $F_{\underline{I}^{**}} \in \mathcal{C}_n\left(\zeta_1,\zeta_2\right)$ such that

$$\underline{TVaR}_{\kappa}^{(\zeta_1,\zeta_2)} = b \times TVaR_{\kappa}\left(I_1^{**} + \ldots + I_n^{**}\right) \text{ for all } \kappa \in [0,1]$$

Dependence uncertainty for exchangeable Bernoulli rvs

Numerical approach = moments bounds

We propose a numerical approach to compute $\underline{TVaR}_{\kappa}^{(\zeta_1,\zeta_2)}$ and $\overline{TVaR}_{\kappa}^{(\zeta_1,\zeta_2)}$

$\underline{TVaR}_{\kappa}^{(\zeta_1,\zeta_2)}$ and $\overline{TVaR}_{\kappa}^{(\zeta_1,\zeta_2)}$ = *moment bounds* are such that

- $\underline{TVaR}_{\kappa}^{(\zeta_1,\zeta_2)} = TVaR_{\kappa}\left(I_1^{**} + \ldots + I_n^{**}\right)$ where $F_{\underline{I}^{**}} \in C_n\left(\zeta_1\right)$
- $\overline{TVaR}_{\kappa}^{(\zeta_1,\zeta_2)} = TVaR_{\kappa}\left(I_1^{**} + \ldots + I_n^{**}\right)$ where $F_{\underline{I}^{**}} \in C_n\left(\zeta_1\right)$

Comments on the approach:

- Range of values for $TVaR_{\kappa}(S_n)$ given partial information on the distribution for $\underline{I}$
- Based on DeFinetti's Representation Theorem
- Inspired from Courtois and Denuit (2009)

Dependence uncertainty for exchangeable Bernoulli rvs

Additional definitions for the approach

Fix: $\zeta_1 = q \in [0,1]$ and ζ_2 such that $\rho_P(I_{i_1}, I_{i_2}) \in [0,1] \cap \left(-\frac{q-q^2}{n-1}, q(1-q)\right)$

Define:

- $\mathcal{D}(\zeta_1; [0,1])$ = class for all F_Θ such that $E[\Theta] = \zeta_1$
- $\mathcal{D}(\zeta_1, \zeta_2; [0,1])$ = class for all F_Θ such that $E[\Theta] = \zeta_1$ and $E\left[\Theta^2\right] = \zeta_2$

Clearly: $C_n(\zeta_1) \Leftrightarrow \mathcal{D}(\zeta_1; [0,1])$ and $C_n(\zeta_1, \zeta_2) \Leftrightarrow \mathcal{D}(\zeta_1, \zeta_2; [0,1])$

Additional definitions for our approach

- $B_l = \left\{\frac{0}{l}, \frac{1}{l}, \frac{2}{l}, \ldots, \frac{l}{l}\right\}$ = support of discrete rv Θ
- $\mathcal{D}(\zeta_1; B_l)$ = class of all F_Θ of $\Theta \in B_l$ such that $E[\Theta] = \zeta_1$
- $\mathcal{D}(\zeta_1, \zeta_2; B_l)$ = class of all F_Θ of $\Theta \in B_l$ such that $E[\Theta] = \zeta_1$ and $E\left[\Theta^2\right] = \zeta_2$

Dependence uncertainty for exchangeable Bernoulli rvs

General steps of the approach

Need 5 steps to find $\underline{TVaR}_{\kappa}^{(\zeta_1,\zeta_2)}$ and $\overline{TVaR}_{\kappa}^{(\zeta_1,\zeta_2)}$

Stop-loss premium of Θ:

- $\pi_{\Theta}\left(\frac{k}{l}\right) = E\left[\max\left(\Theta - \frac{k}{l};0\right)\right] = \sum_{j=k}^{l}\left(1 - F_{\Theta}\left(\frac{j}{l}\right)\right)$, for $k \in \{0,1,\ldots,l-1\}$

Step 1 of 5 : For each $k = 0,1,2,\ldots,l-1$, find $\pi_{\min}\left(\frac{k}{l}\right)$ and $\pi_{\max}\left(\frac{k}{l}\right)$ such that

$$\pi_{\min}\left(\frac{k}{l}\right) \leq \pi_{\Theta}\left(\frac{k}{l}\right) \leq \ \pi_{m,\max}\left(\frac{k}{l}\right)$$

for all $\Theta \in \mathcal{D}\left(\zeta_1,\zeta_2;B_l\right)$

Dependence uncertainty for exchangeable Bernoulli rvs

General steps of the approach

Step 2 of 5: Define two rvs $\Theta^{(\min)}$ and $\Theta^{(\max)}$ with $F_{\Theta^{(\min)}}$ and $F_{\Theta^{(\max)}}$ where

$$\begin{aligned} F_{\Theta^{(\min)}}\left(\frac{k}{l}\right) &= \pi_{\min}\left(\frac{k}{l}\right) - \pi_{\min}\left(\frac{k+1}{l}\right) \\ F_{\Theta^{(\max)}}\left(\frac{k}{l}\right) &= \pi_{\max}\left(\frac{k}{l}\right) - \pi_{\max}\left(\frac{k+1}{l}\right) \end{aligned}$$

for $k = 0,1,2,\ldots,l-1$ and $F_{\Theta^{(\min)}}(1) = F_{\Theta^{(\max)}}(1) = 1$

Step 3 of 5: It implies that

$$\Theta^{(\min)} \preceq_{icx} \Theta \preceq_{icx} \Theta^{(\max)}$$

for all

$$F_{\Theta} \in \mathcal{D}(\zeta_1,\zeta_2;B_l),$$

Note: "$\preceq_{icx}$" means the increasing convex order

Dependence uncertainty for exchangeable Bernoulli rvs

General steps of the approach

Step 4 of 5:

- Let $N_n^{(\min)}$ be defined with $\Theta^{(\min)}$ and $N_n^{(\max)}$ be defined with $\Theta^{(\max)}$
- If

$$\Theta^{(\min)} \preceq_{icx} \Theta \preceq_{icx} \Theta^{(\max)},\ F_\Theta \in \mathcal{D}\left(\zeta_1,\zeta_2;B_l\right),$$

  then

$$N_n^{(\min)} \preceq_{icx} N_n \preceq_{icx} N_n^{(\max)},\ F_{\underline{I}} \in \mathcal{C}_n^+\left(\zeta_1,\zeta_2\right)$$

Step 5 of 5: From Denuit et al. (2005), it follows that

$$\underline{TVaR}_\kappa^{(\zeta_1,\zeta_2)} = TVaR_\kappa\left(bN_n^{(\min)}\right) \leq TVaR_\kappa\left(bN_n\right) \leq TVaR_\kappa\left(bN_n^{(\max)}\right) = \overline{TVaR}_\kappa^{(\zeta_1,\zeta_2)}$$

for all $\kappa \in (0,1)$ and for all $F_{\underline{I}} \in \mathcal{C}_n^+\left(\zeta_1,\zeta_2\right)$

Dependence uncertainty for exchangeable Bernoulli rvs

Numerical Illustration

Amount of individual loss: $b = 1$

Size of the portfolio: number of risks $= n = 1000$ $(l = 100)$

Information about the distribution of $\underline{I}$:

- $q = \zeta_1 = 0.049 = \Pr(\text{occurrence of 1 loss})$ for a risk
- $\zeta_2 = 0.00313 \in (0.049^2, 0.049) \Leftrightarrow \rho_P(I_1, I_2) = 0.01564411 \in (0,1)$

Probabilities: $\zeta_k = \mathcal{L}_Y\left(k \times \mathcal{L}_Y^{-1}(\zeta_1)\right)$

- Case #1: $Y \sim Gamma\left(\frac{1}{\alpha}, 1\right)$: $r = 1.660143$ and $\alpha = 3.082581$
- Case #2: $Y \sim Stable(\frac{1}{\alpha}, 1)$: $r = 2.977504$ and $\alpha = 1.011754$
- Case #3: Fréchet copula with $\alpha = \frac{\zeta_2 - \zeta_1^2}{\zeta_1 - \zeta_1^2} = 0.015644$

Dependence uncertainty for exchangeable Bernoulli rvs

Numerical Illustration

Size of the portfolio: number of risks = $n = 1000$ $(l = 100)$

Risk measure: $\varphi_\kappa(S_n) = TVaR_\kappa(S_n)$

κ	$\frac{\varphi_\kappa(S_n^{\perp})}{n}$	$\frac{\varphi_\kappa\left(S_n^{\min}\right)}{n}$	$\frac{\varphi_\kappa(S_n^{sta})}{n}$	$\frac{\varphi_\kappa(S_n^{gam})}{n}$	$\frac{\varphi_\kappa\left(S_n^{fre}\right)}{n}$	$\frac{\varphi_\kappa(S_n^{\max})}{n}$	$\frac{\varphi_\kappa(S_n^{+})}{n}$
0	0.049	0.049	0.049	0.049	0.049	0.049	0.049
0.5	0.0544	0.0556	0.0569	0.0684	0.0558	0.0762	0.098
0.9	0.0613	0.0655	0.0806	0.1044	0.0685	0.1312	0.49
0.99	0.0681	0.0792	0.2243	0.1537	0.1398	0.3146	1
0.999	0.0735	0.0886	0.7199	0.2006	0.7844	0.8942	1

Conclusion

For fixed ζ_1 and ζ_2, the multivariate mixed beta-Bernoulli distribution is frequently chosen

We have briefly presented (recalled) few approaches to define $F_{\underline{I}}$ and F_{N_n}

Dependence uncertainty = model risk: we have presented an approach to find moments bounds on TVaR (for fixed ζ_1 and ζ_2)

The approach is valid for the expectation of any convex function of N_n

Moment bounds can be improved by adding constraints on the third moment of Θ

Thank you for your attention !

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