Cellular Automata as an Alternative to Geometric Brownian Motion

Frank Cuypers

Outline

- Geometric Brownian Motion
- Cellular Automata

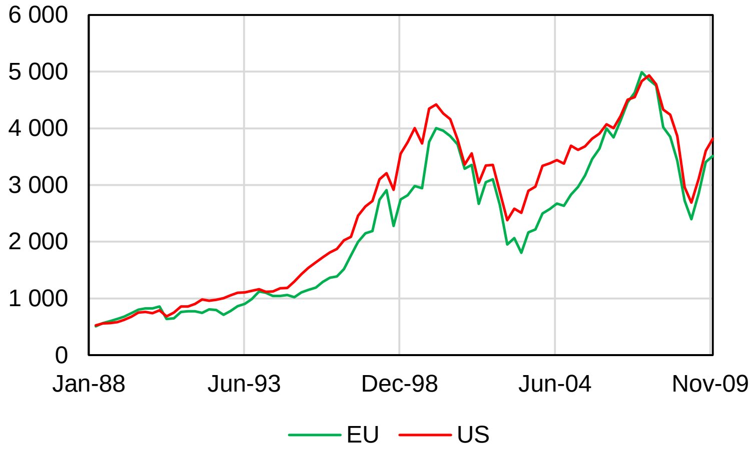
Outline

- Geometric Brownian Motion
- Cellular Automata

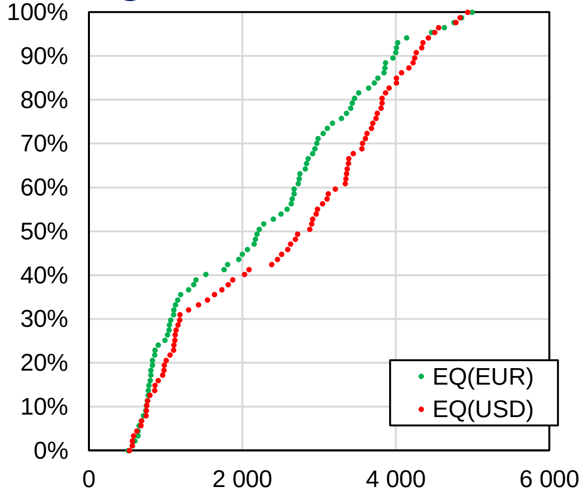
Choice of Variables

2 equity indices: EU & US

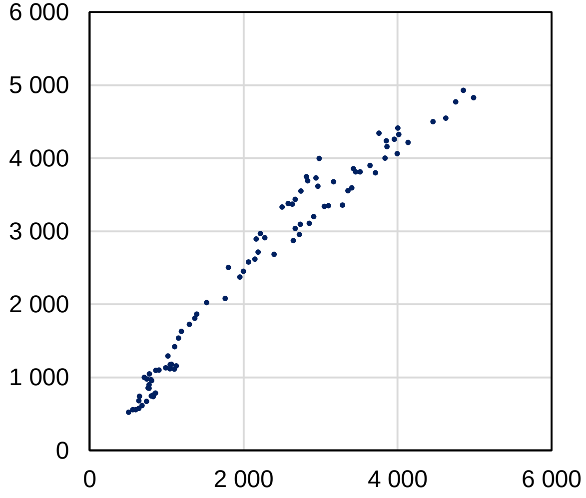
- Index values = X_t = auxiliary derived variables
  - ⇒ Inappropriate for modelling



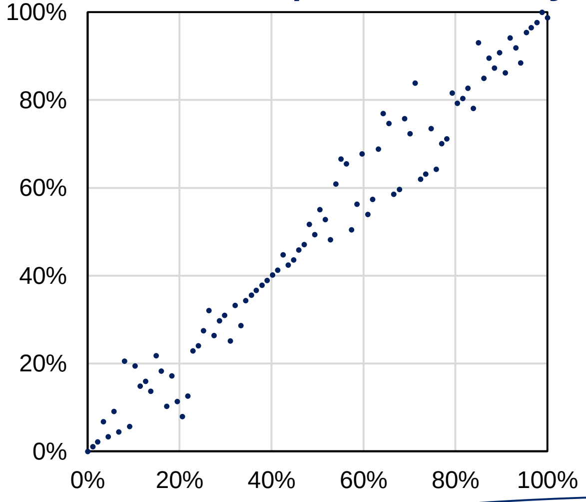
marginals CDF



linear correlation ~ 98%



copula density



Choice of Variables

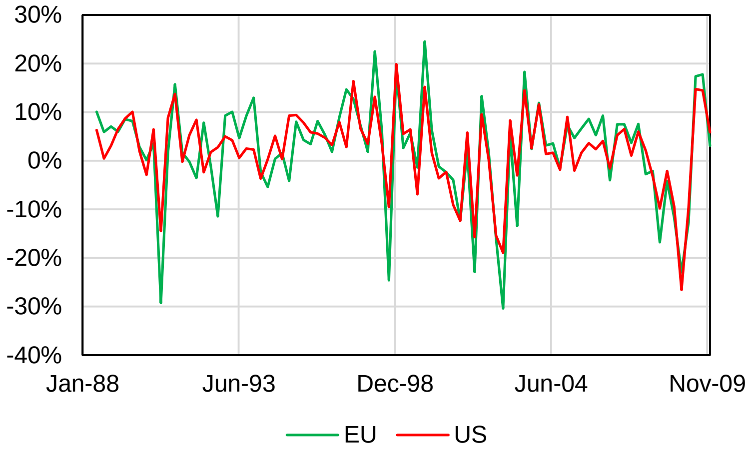
2 equity indices: EU & US

- Index values = X_t = auxiliary derived variables

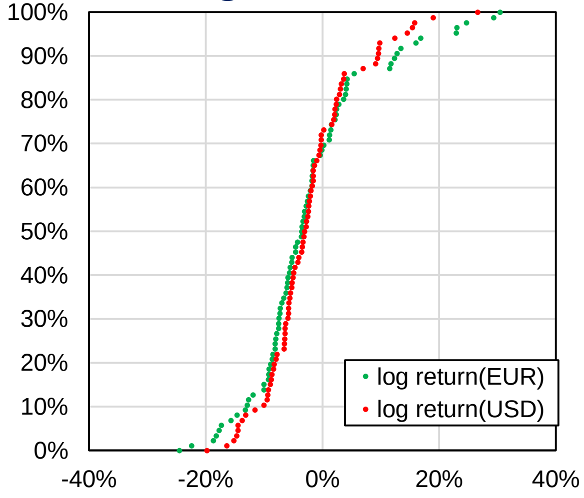
  ⇒ Inappropriate for modelling

- Log returns $r_t = \ln \frac{X_t}{X_{t-1}}$ = fundamental stationary variables

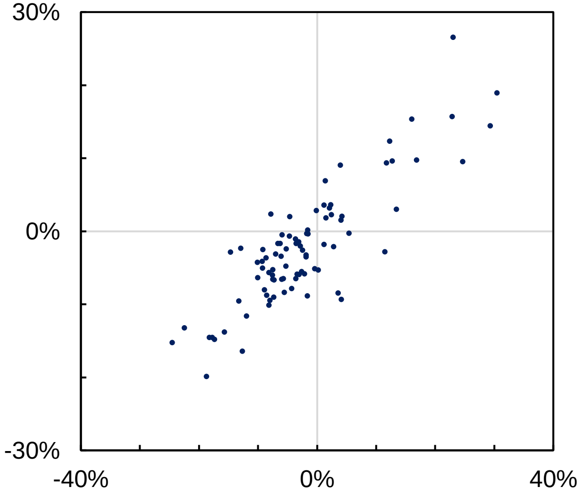
  ⇒ Easier to model



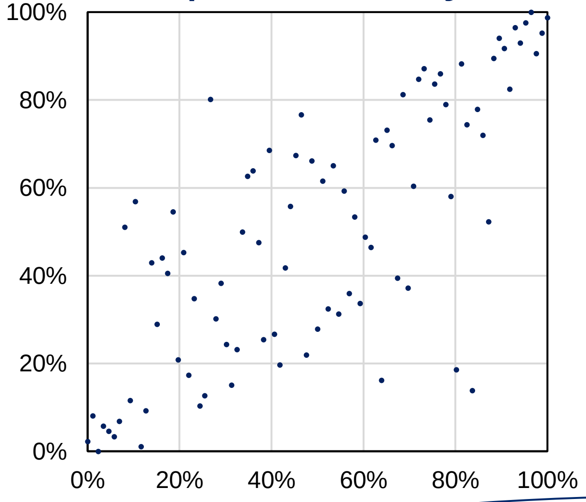
marginals CDF



linear correlation ~ 78%



copula density



Geometric Brownian Motion

- Risk-free $dX = \mu X\,dt \quad \Rightarrow \quad X = X_0\, e^{\mu t}$

- GBM $dX = \underbrace{\mu X\,dt}_{\text{drift}} + \underbrace{\sigma X\,dW}_{\text{diffusion}}$

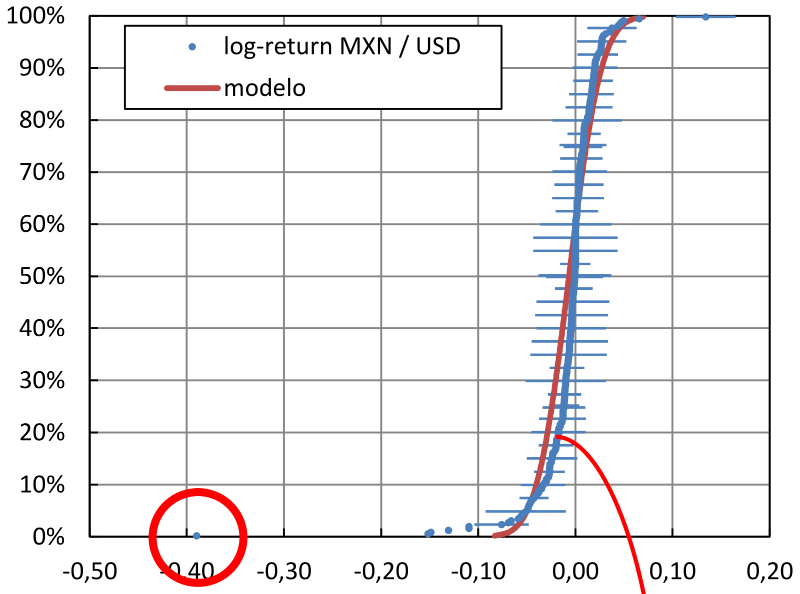
  Wiener process $W \sim N\left(0, \sqrt{t}\right)$

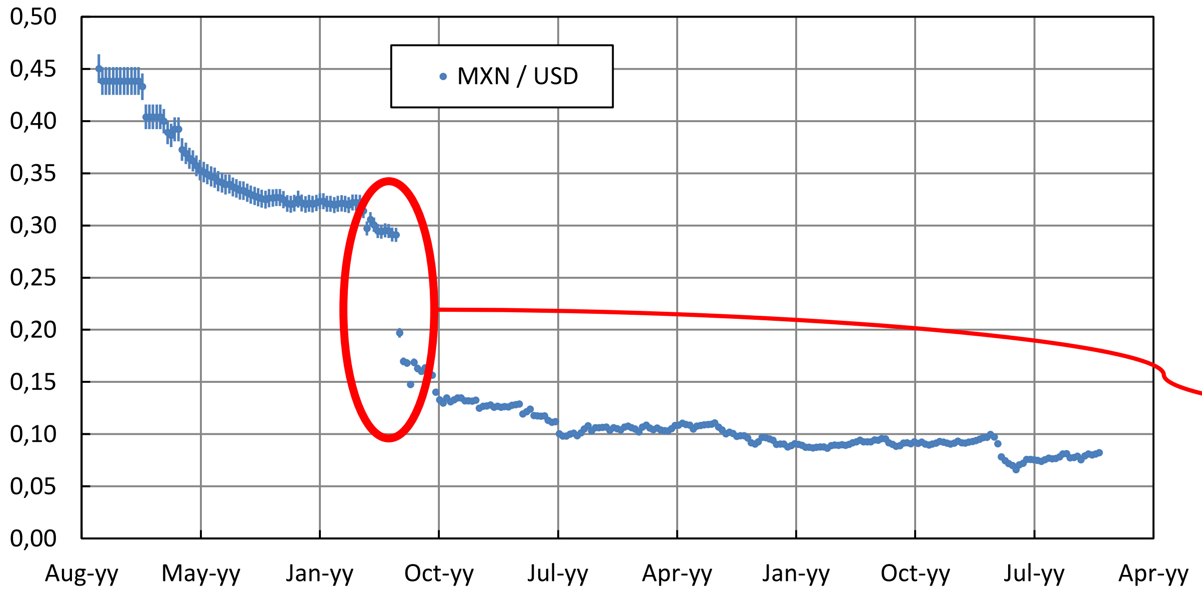
- Logreturn (Itô) $r_t = \ln \frac{X_t}{X_0} \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right) t\,,\ \sigma\sqrt{t}\right) \quad \Rightarrow \quad X_t \sim$ lognormal

- Statistics
  $$\mathbb{E}[X_t] = X_0\, e^{\mu t}$$
  $$\mathbb{V}[X_t] = \mathbb{E}[X_t]^2 \left(e^{\sigma^2 t} - 1\right)$$

Model Risk

normal models 💣





χ^2 / dof = 0.6

☹

Challenges with Geometric Brownian Motion

- No shocks

$\rightarrow$ Non-normal log returns	BUT	scaling
$\rightarrow$ Lévy process	BUT	more complex
$\rightarrow$ Regime-switching lognormal models	BUT	more parameters

- No heteroskedasticity

$\rightarrow$ Stochastic volatility models (Heston)	BUT	tough to calibrate
$\rightarrow$ GARCH process	BUT	more parameters

Outline

- Geometric Brownian Motion
- Cellular Automata

Cellular Automaton

Nearest neighbour 2 states XOR automaton:

$$Z_i(t) \;=\; Z_{i-1}(t-1) \;\oplus\; Z_{i+1}(t-1)$$

⊕	-1	1
-1	-1	1
1	1	-1

t	Z_1	Z_2	Z_3	Z_4	Z_5	Z_1
0	-1	-1	1	-1	1	-1
1						
2						
3						
4						

Cellular Automaton

Nearest neighbour 2 states XOR automaton:

$$Z_i(t) = Z_{i-1}(t-1) \oplus Z_{i+1}(t-1)$$

$\oplus$	**-1**	**1**
-1	-1	1
1	1	-1

t	Z_1	Z_2	Z_3	Z_4	Z_5	Z_1
0	-1	-1	1	-1	1	-1
1		1				
2						
3						
4						

Cellular Automaton

Nearest neighbour 2 states XOR automaton:

$$Z_i(t) = Z_{i-1}(t-1) \oplus Z_{i+1}(t-1)$$

$\oplus$	-1	1
-1	-1	1
1	1	-1

t	Z_1	Z_2	Z_3	Z_4	Z_5	Z_1
0	-1	-1	1	-1	1	-1
1	-1	1	-1	-1	-1	-1
2						
3						
4						

Cellular Automaton

Nearest neighbour 2 states XOR automaton:

$$Z_i(t) = Z_{i-1}(t-1) \oplus Z_{i+1}(t-1)$$

$\oplus$	-1	1
-1	-1	1
1	1	-1

t	Z_1	Z_2	Z_3	Z_4	Z_5	Z_1
0	-1	-1	1	-1	1	-1
1	-1	1	-1	-1	-1	-1
2	1	-1	1	-1	-1	1
3	1	-1	-1	1	1	1
4	1	1	1	1	-1	1

Cellular Automaton

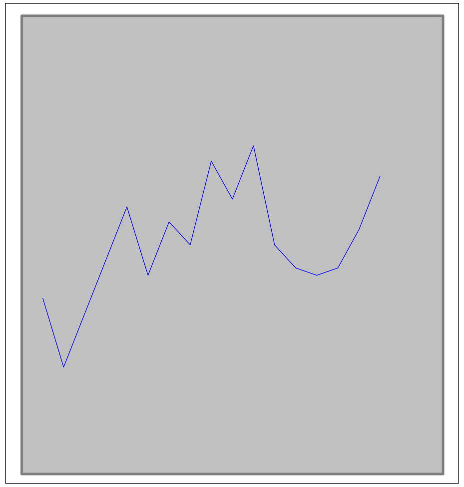
⊕	-1	1
-1	-1	1
1	1	-1

Nearest neighbour 2 states XOR automaton:

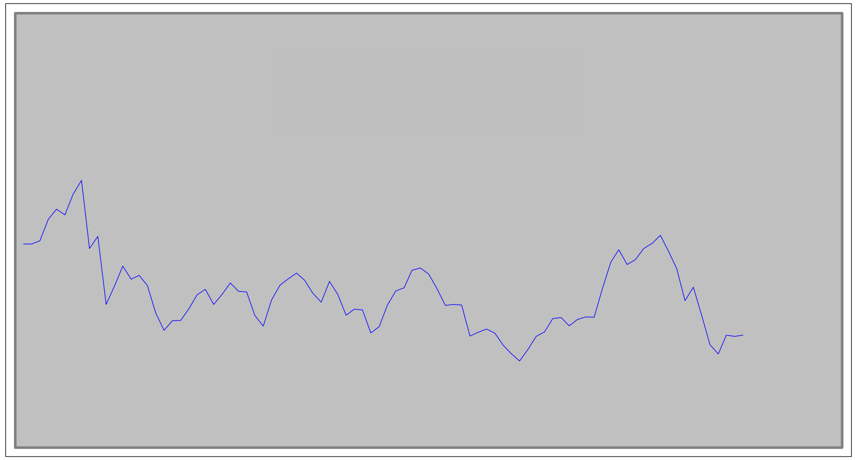
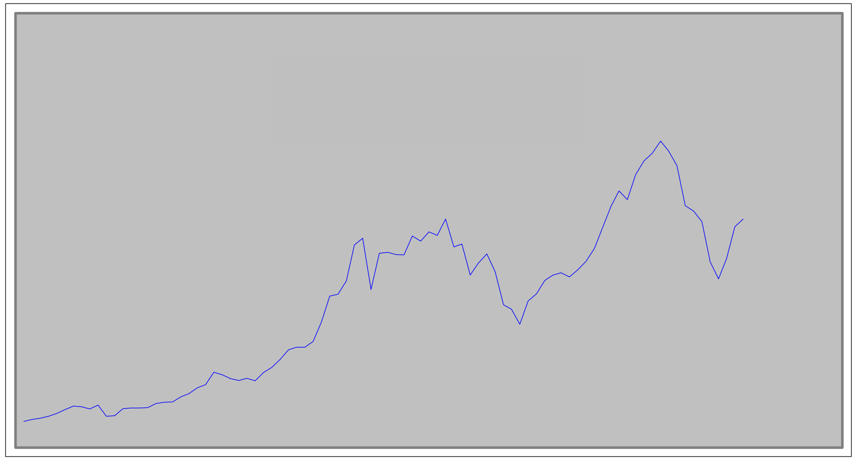
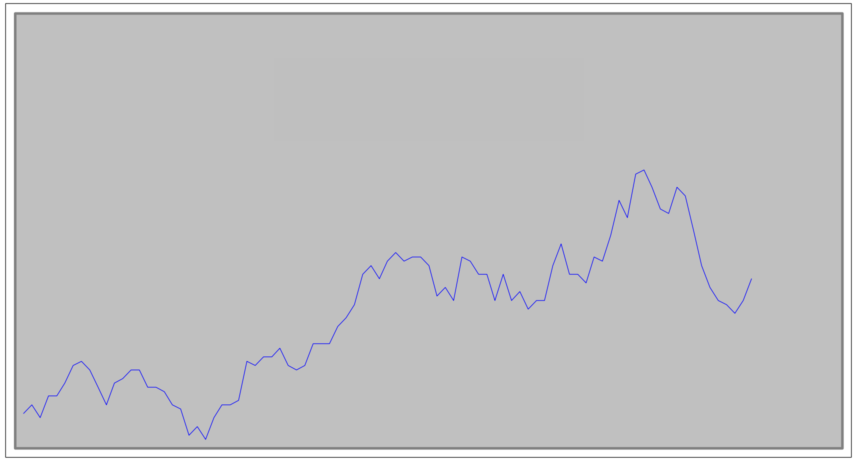
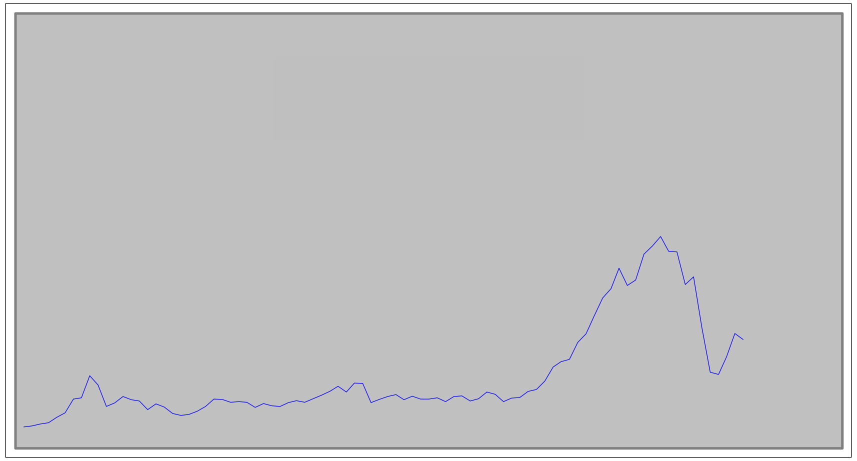
$$Z_i(t) = Z_{i-1}(t-1) \oplus Z_{i+1}(t-1)$$

$$I(t) = I(t-1) + \sum_{i=1}^{N} Z_i(t)$$

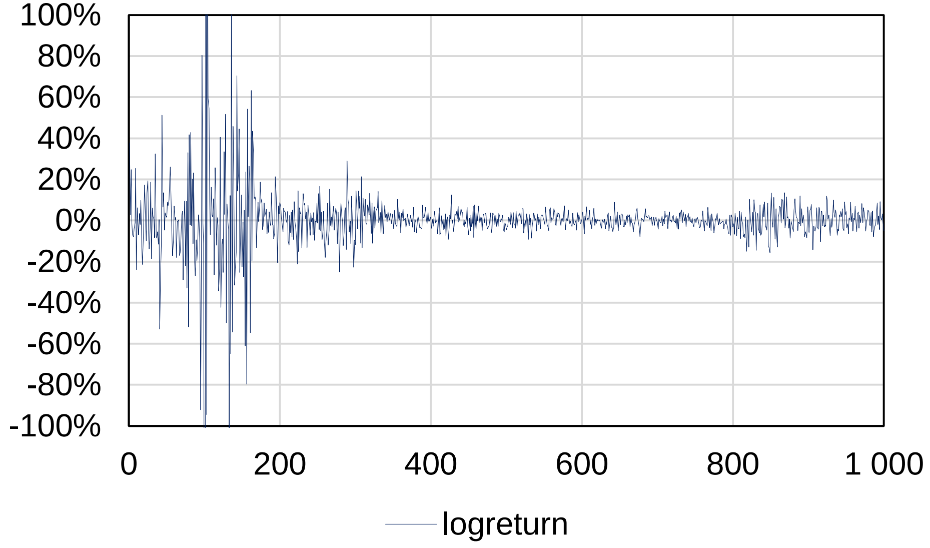
t	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8	Z_9	Z_{10}	Z_{11}	Z_{12}	Z_{13}	Z_{14}	Z_{15}	Z_{16}	Z_{17}	Z_{18}	Z_{19}	Z_{20}	Z_{21}	Z_{22}	Z_{23}	Z_{24}	Z_{25}	Z_{26}	Z_{27}	Z_{28}	Z_{29}	Z_{30}	Z_{31}	Z_{32}	Z_{33}	Z_{34}	Z_{35}	Z_{36}	Z_{37}	Z_{38}	Z_{39}	Z_{40}	Z_1
0	1	-1	1	1	1	1	-1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	-1	1	-1	-1	-1	-1	1	1	1	1	1	-1	1	-1	-1	-1	1	-1	1	1
1	1	-1	1	-1	-1	1	-1	1	1	1	-1	-1	-1	-1	1	1	-1	-1	-1	-1	1	-1	-1	1	-1	-1	1	1	-1	-1	-1	1	-1	-1	1	-1	1	-1	-1	1	1
2	1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	-1	1	1	1	1	-1	-1	1	-1	1	1	-1	1	1	1	1	1	-1	1	-1	1	1	-1	-1	-1	1	1	1	1
3	1	1	1	1	1	1	1	-1	-1	1	1	1	1	1	-1	-1	1	1	1	-1	-1	1	1	-1	1	-1	-1	-1	1	-1	-1	-1	1	1	1	-1	1	1	-1	-1	1
4	-1	-1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	1	1	1	1	-1	1	1	1	1	1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1	1	1	1	1	-1
5	-1	-1	-1	-1	-1	1	1	-1	-1	1	1	-1	1	1	-1	-1	1	-1	1	-1	-1	-1	1	1	1	-1	-1	-1	-1	-1	-1	1	1	-1	-1	-1	1	-1	-1	1	-1
6	-1	-1	-1	-1	1	1	1	1	1	1	1	-1	1	1	1	1	-1	-1	-1	1	-1	1	1	-1	1	1	-1	-1	-1	-1	1	1	1	1	-1	1	-1	1	1	-1	-1
7	-1	-1	-1	1	1	-1	-1	-1	-1	-1	1	-1	1	-1	-1	1	1	-1	1	-1	-1	1	1	-1	1	1	1	-1	-1	1	1	-1	-1	1	-1	-1	-1	1	1	1	-1
8	-1	-1	1	1	1	1	-1	-1	-1	1	-1	-1	-1	1	1	1	1	-1	-1	1	1	1	1	-1	1	-1	1	1	1	1	1	1	1	-1	1	-1	1	1	-1	1	-1
9	-1	1	1	-1	-1	1	1	-1	1	-1	1	-1	1	1	-1	-1	1	1	1	1	-1	-1	1	-1	-1	-1	1	-1	-1	-1	-1	-1	1	-1	-1	-1	1	1	-1	-1	-1
10	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	1	1	1	1	1	-1	-1	1	1	1	-1	1	-1	1	-1	1	-1	-1	-1	1	-1	1	-1	1	1	1	1	-1	1
11	-1	-1	-1	-1	-1	-1	1	1	-1	-1	-1	1	1	-1	-1	-1	1	1	1	1	-1	1	-1	-1	-1	-1	-1	-1	1	-1	1	-1	-1	-1	-1	1	-1	-1	1	-1	-1
12	-1	-1	-1	-1	-1	1	1	1	1	-1	1	1	1	1	-1	1	1	-1	-1	1	-1	-1	1	-1	-1	-1	-1	1	-1	-1	-1	1	-1	-1	1	-1	1	1	-1	1	-1
13	-1	-1	-1	-1	1	1	-1	-1	1	-1	1	-1	-1	1	-1	1	1	1	1	-1	1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	-1	1	1	-1	-1	-1
14	-1	-1	-1	1	1	1	1	1	-1	-1	-1	1	1	-1	-1	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1	-1	-1
15	-1	-1	1	1	-1	-1	-1	1	1	-1	1	1	1	1	1	-1	1	1	-1	-1	1	1	1	1	1	1	1	-1	-1	-1	-1	1	1	-1	-1	-1	-1	-1	1	1	-1
16	-1	1	1	1	1	-1	1	1	1	-1	1	-1	-1	-1	1	-1	1	1	1	1	1	-1	-1	-1	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1	-1	1	1	1	-1

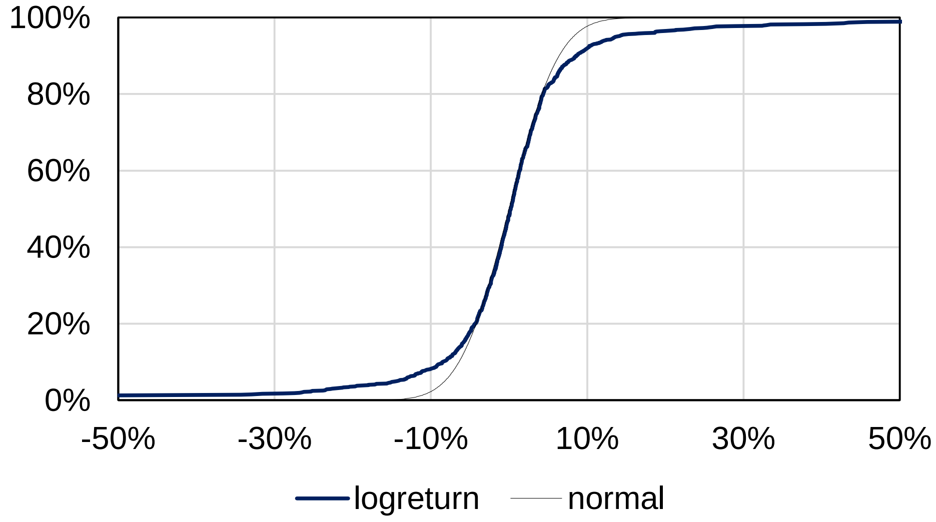


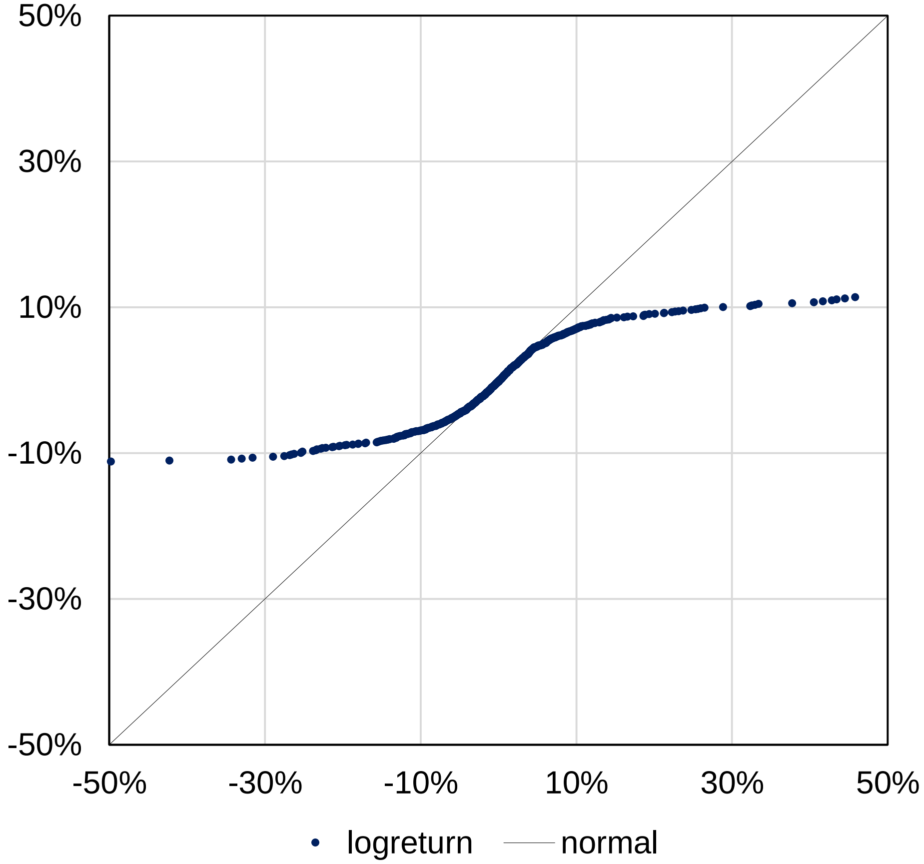
Equity Indices 1988 – 2010

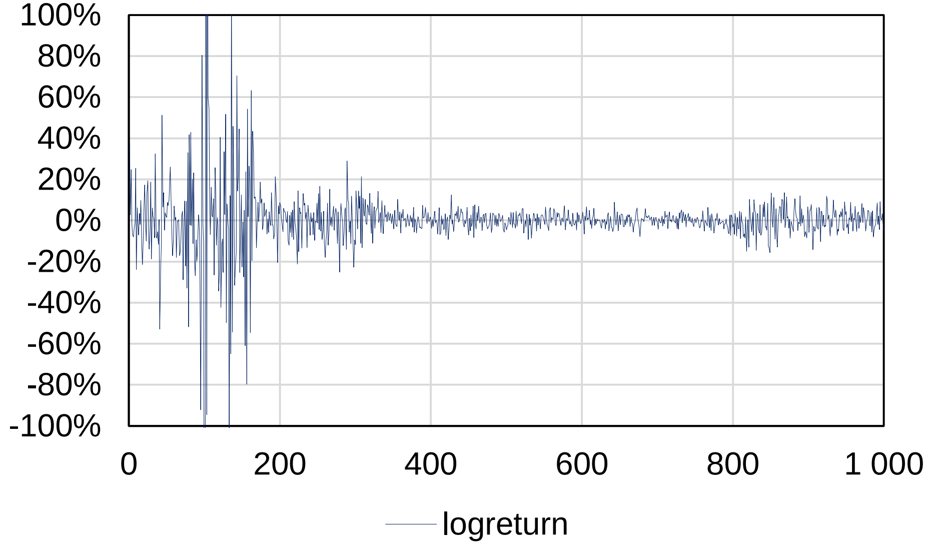
Logreturns: Distribution

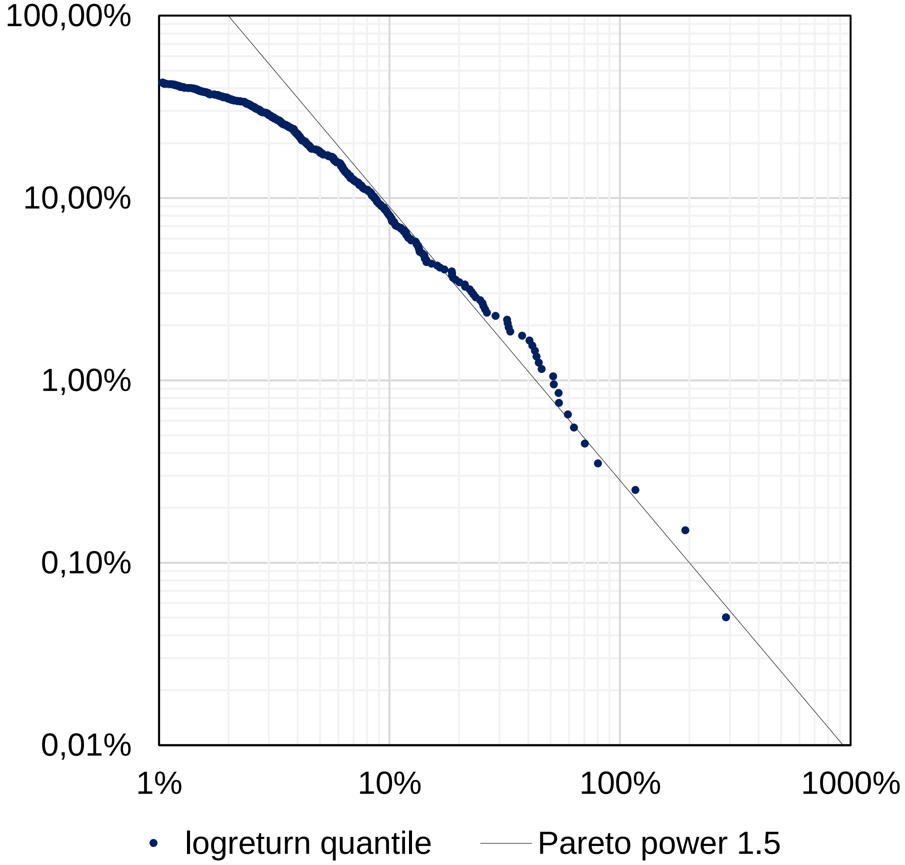




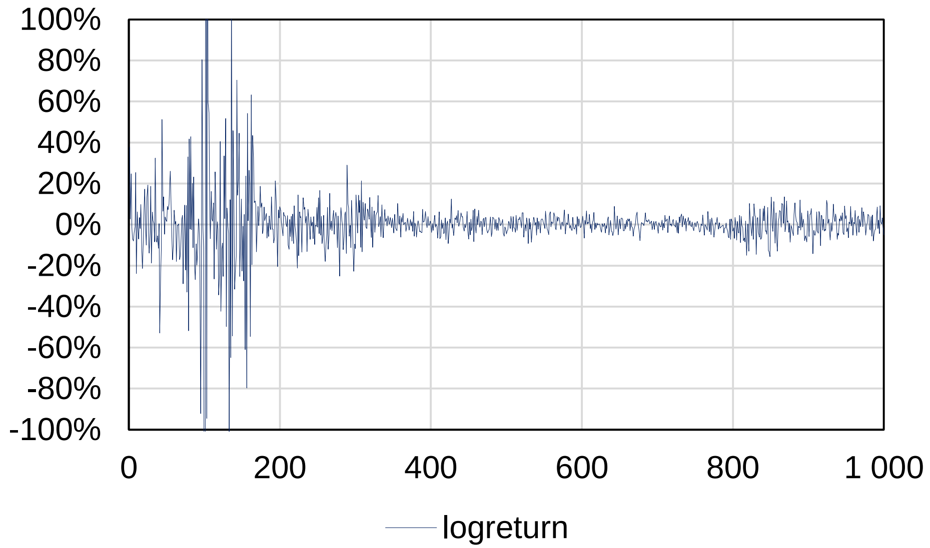


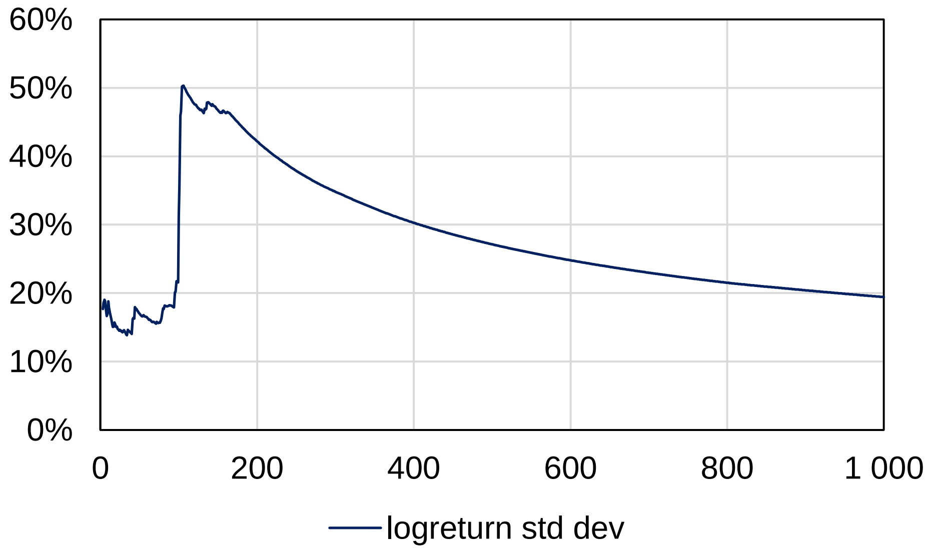
Logreturns: Tail



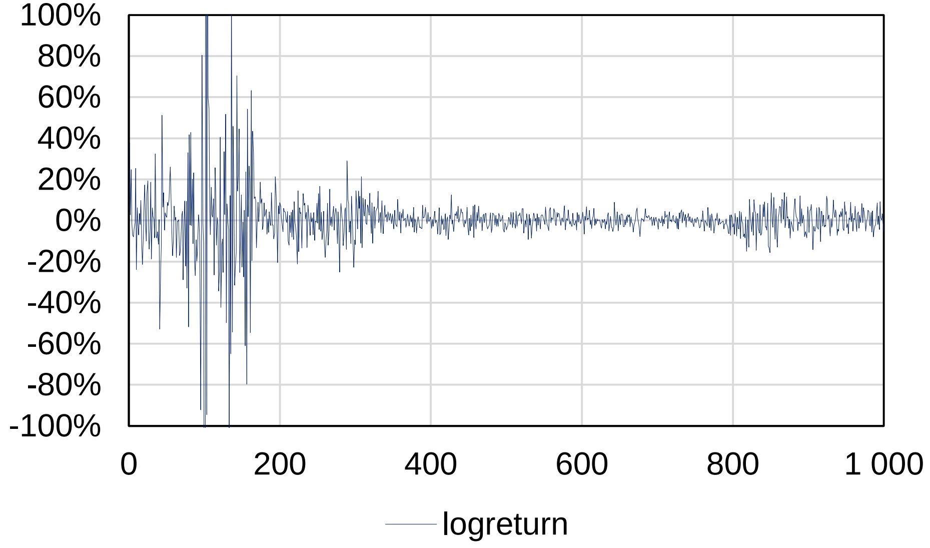


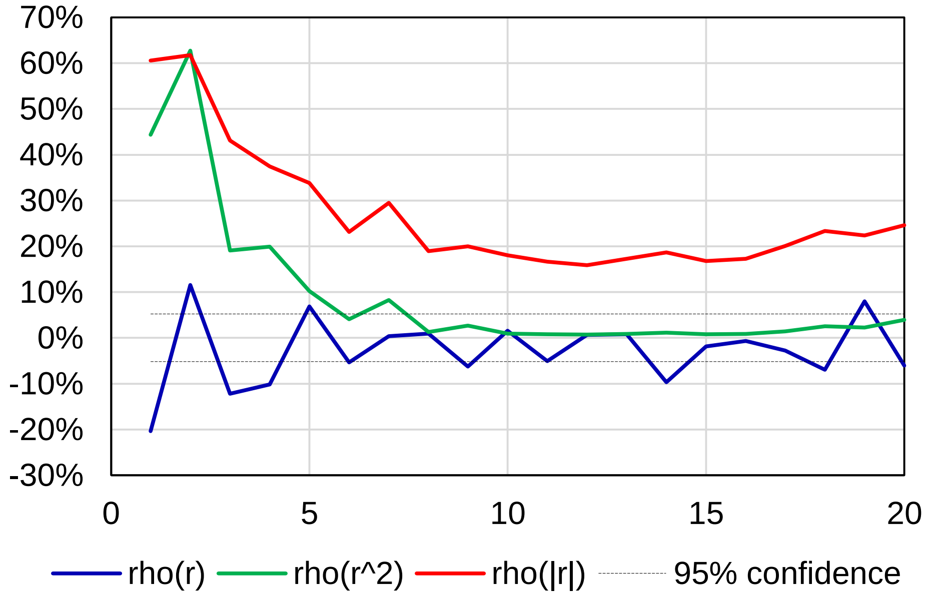
Logreturns: Volatility

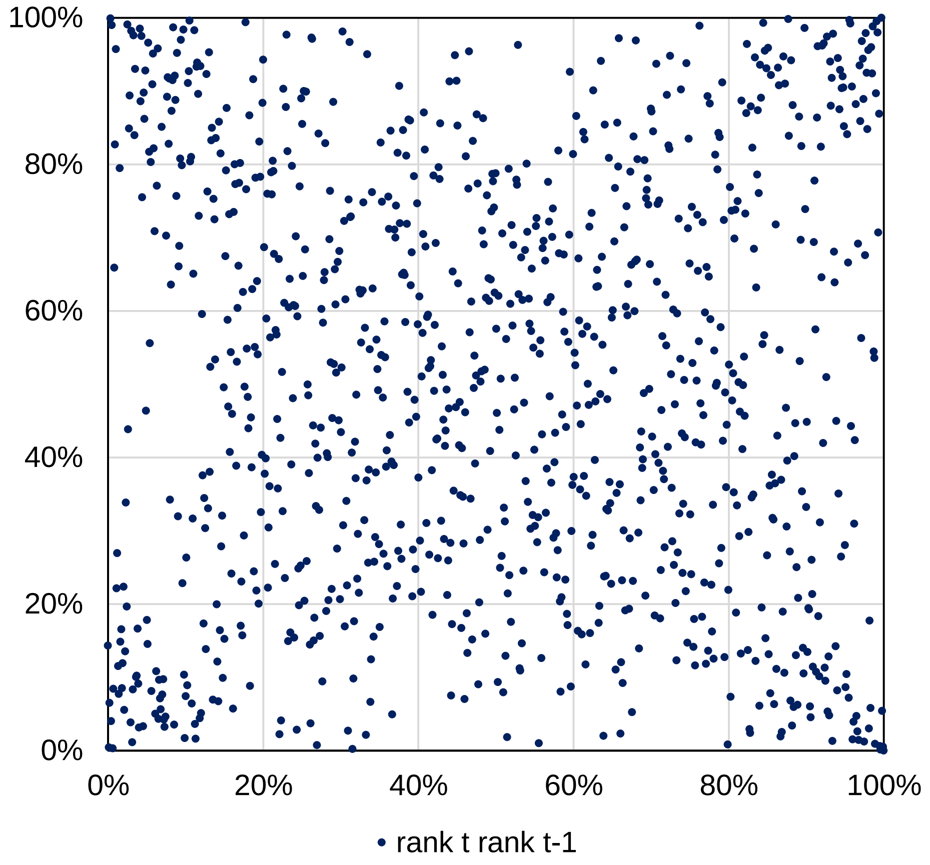




Logreturns: Autocorrelation







Outline

- Geometric Brownian Motion
- Cellular Automata

Challenges with Geometric Brownian Motion

- No shocks

→ Non-normal log returns	BUT	scaling
→ Lévy process	BUT	more complex
→ Regime-switching lognormal models	BUT	more parameters

- No heteroskedasticity

→ Stochastic volatility models (Heston)	BUT	tough to calibrate
→ GARCH process	BUT	more parameters

- Rule 90 cellular automaton
  - Reproduces the rich phenomenology of financial time series
  - Requires minimal amount of parameters

Lecturer’s Coordinates

Frank Cuypers

+41 (41) 725 32 94

frank.cuypers@prs-zug.com