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A Fuzzy Least Squares Estimation of a Hybrid log-Poisson Regression and its Goodness of Fit for Optimal Loss Reserves in Insurance

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Problem statement

- In loss reserving, the log-Poisson regression model is a well known as a stochastic model underlying the Chain-Ladder method which is the most used method for reserving purposes Mack (1991).
- So in order to improve the Chain Ladder method, one can just improve the log-Poisson regression model in loss reserving framework.
- Straub and Swiss, (1988) proved that one can face fuzzy data in loss reserving, for example when the claims are related to body injures.
- Thus, one estimate a hybrid generalized linear model (log-Poisson) using the fuzzy least-squares (FLS) procedures (Celmiņš, 987a,b; D'Urso and Gastaldi, 2000; DUrso and Gastaldi, 2001) to handle both fuzziness and randomness in loss reserving.

Problem statement

- We develop a new goodness of fit index to compare this new model and the classical log-Poisson regression (Mack, 1991).
- Both the classical log-Poisson model and the hybrid one are performed on a *run-off triangle* data.
- According to the goodness of fit index and the Mean Square Error Prediction(MSEP), one can prove that the new model provide better results than the classical log-Poisson model, hence the Chain Ladder method.

The New Hybrid model

- In this subsection, we present the new hybrid log-Poisson regression, which is an extension of the log-Poisson regression (Mack, 1991) in loss reserving framework.
- Mack (1991) assumes that the incremental payments Y_{ij} are log-Poisson distributed, i.e,

$$Y_{ij} \sim \mathcal{P}(e^{\nu_{ij}}) \Rightarrow \mathbb{E}(Y_{ij}) = e^{\nu_{ij}} = \varphi_{ij} \qquad (1)$$
$$\forall (i,j) \in \{i = 1, \dots, k\} \times \{j = 1, \dots, k - i + 1\}.$$

- Now we assume that uncertainty about Y_{ij} in the *run-off triangle* is due both to fuzziness and randomness. Then one can suppose that $\tilde{Y}_{ij} = (Y_{ij}^L, Y_{ij}^c, Y_{ij}^R)$ is a fuzzy Poisson random variable (Buckley, 2006), i.e,

$$\begin{aligned}\left[\mathbb{E}_F(\tilde{Y}_{ij})\right]_h &= \{\sum_{x=0}^{+\infty} x e^{-\varphi_{ij}} \frac{(\varphi_{ij})^x}{x!} \mid \varphi_{ij} \in [\tilde{Y}_{ij}]_h\} \\ &= \{\varphi_{ij} \mid \varphi_{ij} \in [\tilde{Y}_{ij}]_h\} \\ &= \tilde{\varphi}_{ij},\end{aligned}$$

where $\mathbb{E}_F(\cdot)$ is the fuzzy expected value operator. So the fuzzy expected value is just the fuzzification of the crisp expected value.

The New Hybrid model

The hybrid model built over the log-Poisson regression can be defined in matrix form as follows:

$$\begin{cases} \ln(Y^c) & = Y^{c^*} + \ln(\boldsymbol{\varepsilon}),\ Y^{c^*} = \mathbf{X}\boldsymbol{\beta} \\ \ln(Y^L) & = Y^{L^*} + \ln(\boldsymbol{\xi}),\ Y^{L^*} = Y^{c^*}\theta + \mathbf{1}\lambda \\ \ln(Y^R) & = Y^{R^*} + \ln(\boldsymbol{\eta}),\ Y^{R^*} = Y^{c^*}\delta + \mathbf{1}\mu \end{cases}$$

$$\Leftrightarrow$$

$$\begin{cases} Y^{c'} & = Y^{c^*} + \boldsymbol{\varepsilon}',\ Y^{c^*} = \mathbf{X}\boldsymbol{\beta} \\ Y^{L'} & = Y^{L^*} + \boldsymbol{\xi}',\ Y^{L^*} = Y^{c^*}\theta + \mathbf{1}\lambda \\ Y^{R'} & = Y^{R^*} + \boldsymbol{\eta}',\ Y^{R^*} = Y^{c^*}\delta + \mathbf{1}\mu \end{cases} \tag{2}$$

where

- $\boldsymbol{\beta} = \left(\tau, \boldsymbol{\alpha}, \boldsymbol{\gamma}\right)^T \in \mathbb{R}^{2k-1}$ with

$$\tau \in \mathbb{R}$$
$$\boldsymbol{\alpha} = (\alpha_2 \ \ldots \ \alpha_k) \in \mathbb{R}^{k-1}$$
$$\boldsymbol{\gamma} = (\gamma_2 \ \ldots \ \gamma_k) \in \mathbb{R}^{k-1}$$

- $$Y^{c'} = \ln(Y^c); \qquad Y^{L'} = \ln(Y^L); \qquad Y^{R'} = \ln(Y^R)$$
$$\boldsymbol{\varepsilon}' = \ln(\boldsymbol{\varepsilon}); \qquad \boldsymbol{\xi}' = \ln(\boldsymbol{\xi}); \qquad \boldsymbol{\eta}' = \ln(\boldsymbol{\eta}).$$

- $\boldsymbol{\varepsilon}, \boldsymbol{\xi}, \boldsymbol{\eta}$ are $n \times 1$ vectors of uncorrelated error terms following Poisson random variables ($\mathcal{P}(1)$) such that $\mathbb{E}(\boldsymbol{\varepsilon}') = \mathbb{E}(\boldsymbol{\xi}') = \mathbb{E}(\boldsymbol{\eta}') = 0_{n\times 1}$.

The New Hybrid model

Theorem

The iterative FLS estimators $\hat{\beta}, \hat{\theta}, \hat{\delta}, \hat{\lambda}$ *and* $\hat{\mu}$ *of the parameters* $\beta, \theta, \delta, \lambda$ *and* μ *in model* (2) *are given by :*

$$\hat{\beta} = \frac{1}{\left(1+\theta^2+\delta^2\right)}(\mathbf{X}^T\mathbf{X})^{-1}\Big\{\mathbf{X}^T Y^{c'} + \theta\mathbf{X}^T\left(Y^{L'} - \mathbf{1}\lambda\right) + \delta\mathbf{X}^T\left(Y^{R'} - \mathbf{1}\mu\right)\Big\} \tag{3}$$

$$\hat{\theta} = \left[\beta^T(\mathbf{X}^T\mathbf{X})\beta\right]^{-1}\beta^T\mathbf{X}^T\left(Y^{L'} - \mathbf{1}\lambda\right) \tag{4}$$

$$\hat{\delta} = \left[\beta^T(\mathbf{X}^T\mathbf{X})\beta\right]^{-1}\beta^T\mathbf{X}^T\left(Y^{R'} - \mathbf{1}\mu\right) \tag{5}$$

$$\hat{\lambda} = \frac{1}{n}\left(\mathbf{1}^T Y^{L'} - \beta^T\mathbf{X}^T\mathbf{1}\theta\right) \tag{6}$$

$$\hat{\mu} = \frac{1}{n}\left(\mathbf{1}^T Y^{R'} - \beta^T\mathbf{X}^T\mathbf{1}\delta\right) \tag{7}$$

where $\beta, \theta, \delta, \lambda$ *and* μ *are the different values taken by the parameters before reaching their optimal values* $\hat{\beta}, \hat{\theta}, \hat{\delta}, \hat{\lambda}$ *and* $\hat{\mu}$*.*

Proof:(*see the manuscript*)

The goodness of fit index

- In this subsection, one derive a goodness of fit index $\tilde{R}_F^2$ for the hybrid log-Poisson regression. This index is relevant to assess the explanatory power of the model.
- In order to provide the mathematical formula of that $\tilde{R}_F^2$, let's present some results.

Proposition

Consider the following hybrid model

$$\begin{cases} Y^{c'} & = Y^{c^*} + \boldsymbol{\varepsilon}', \; Y^{c^*} = \mathbf{X}\beta \\ Y^{L'} & = Y^{L^*} + \boldsymbol{\xi}', \; Y^{L^*} = Y^{c^*}\theta + \mathbf{1}\lambda \\ Y^{R'} & = Y^{R^*} + \boldsymbol{\eta}', \; Y^{R^*} = Y^{c^*}\delta + \mathbf{1}\mu \end{cases} \tag{8}$$

where

- $\boldsymbol{\beta} = \left(\tau, \boldsymbol{\alpha}, \boldsymbol{\gamma}\right)^T \in \mathbb{R}^{2k-1}, \tau \in \mathbb{R}$ *with*

$$\boldsymbol{\alpha} = (\alpha_2 \; \dots \; \alpha_k) \in \mathbb{R}^{k-1}$$

$$\boldsymbol{\gamma} = (\gamma_2 \; \dots \; \gamma_k) \in \mathbb{R}^{k-1}$$

- $\boldsymbol{\varepsilon}, \boldsymbol{\xi}, \boldsymbol{\eta}$ *are* $n \times 1$ *vectors of error terms following Poisson random variables* *(*$\mathcal{P}(1)$*).*

The goodness of fit index

The following relationships hold :

1) $$\sum_{i=1}^{k}\sum_{j=1}^{k-i+1} Y_{ij}^{c^*}\left(Y_{ij}^{c'} - Y_{ij}^{c^*}\right) = 0$$

2)

- $$\sum_{i=1}^{k}\sum_{j=1}^{k-i+1} \left(Y_{ij}^{c'} - Y_{ij}^{c^*}\right) = 0$$
- $$\sum_{i=1}^{k}\sum_{j=1}^{k-i+1} \left(Y_{ij}^{L'} - Y_{ij}^{L^*}\right) = 0$$
- $$\sum_{i=1}^{k}\sum_{j=1}^{k-i+1} \left(Y_{ij}^{R'} - Y_{ij}^{R^*}\right) = 0$$

3)

- $$\sum_{i=1}^{k}\sum_{j=1}^{k-i+1} \left(Y_{ij}^{L'} - Y_{ij}^{L^*}\right)Y_{ij}^{L^*} = 0$$
- $$\sum_{i=1}^{k}\sum_{j=1}^{k-i+1} \left(Y_{ij}^{R'} - Y_{ij}^{R^*}\right)Y_{ij}^{R^*} = 0$$

Proof : (*See the manuscript*)

The goodness of fit index

Definition

For a set of crisp observations $\mathbf{X}_{n\times(p+1)}$ and by considering the hybrid log-Poisson regression built over the classical one (2) in loss reserving framework, one can define for the fuzzy output $\tilde{Y}_{ij} = (Y_{ij}^L, Y_{ij}^c, Y_{ij}^R),\ (i,j) \in \{1,\dots,k\} \times \{1,\dots,k-i+1\}$ the following concepts :

1) The fuzzy total sum of squares

$$FSST = \sum_{i=1}^{k}\sum_{j=1}^{k-i+1}\left(Y_{ij}^{c'} - \overline{Y}_{\ln}^{c}\right)^2 + \sum_{i=1}^{k}\sum_{j=1}^{k-i+1}\left(Y_{ij}^{L'} - \overline{Y}_{\ln}^{L}\right)^2 + \sum_{i=1}^{k}\sum_{j=1}^{k-i+1}\left(Y_{ij}^{R'} - \overline{Y}_{\ln}^{R}\right)^2 \quad (9)$$

The goodness of fit index

where

$$\overline{Y}^{c}_{\ln} = \frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{k-i+1} Y^{c'}_{ij};\ \overline{Y}^{L}_{\ln} = \frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{k-i+1} Y^{L'}_{ij} \qquad (10)$$

$$\overline{Y}^{R}_{\ln} = \frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{k-i+1} Y^{R'}_{ij},\ n = \frac{1}{2}k(k+1) \qquad (11)$$

2) The fuzzy sum of the squares of the regression

$$FSSR = \sum_{i=1}^{k}\sum_{j=1}^{k-i+1}\left(Y^{c^*}_{ij} - \overline{Y}^{c}_{\ln}\right)^2 + \sum_{i=1}^{k}\sum_{j=1}^{k-i+1}\left(Y^{L^*}_{ij} - \overline{Y}^{L}_{\ln}\right)^2 + \sum_{i=1}^{k}\sum_{j=1}^{k-i+1}\left(Y^{R^*}_{ij} - \overline{Y}^{R}_{\ln}\right)^2 \qquad (12)$$

3) The fuzzy sum of the squares of errors

$$FSSE = \sum_{i=1}^{k}\sum_{j=1}^{k-i+1}\left(Y^{c'}_{ij} - Y^{c^*}_{ij}\right)^2 + \sum_{i=1}^{k}\sum_{j=1}^{k-i+1}\left(Y^{L'}_{ij} - Y^{L^*}_{ij}\right)^2 + \sum_{i=1}^{k}\sum_{j=1}^{k-i+1}\left(Y^{R'}_{ij} - Y^{R^*}_{ij}\right)^2 \qquad (13)$$

The goodness of fit index

Theorem

Let's consider a set of crisp observations $\mathbf{X}_{n\times(p+1)}$ *and fuzzy output* $\tilde{Y}_{ij} = (Y_{ij}^L, Y_{ij}^c, Y_{ij}^R),\ (i,j) \in \{1,\dots,k\} \times \{1,\dots,k-i+1\}$. *By considering the hybrid log-Poisson regression built over the classical one (equation (2)) in loss reserving framework, the following relationship holds :*

$$FSST = FSSR + FSSE \qquad (14)$$

Proof: (*See the manuscript*)

Definition

Let us consider a set of crisp observations $\mathbf{X}_{n\times(p+1)}$ and fuzzy output $\tilde{Y}_{ij} = (Y_{ij}^L, Y_{ij}^c, Y_{ij}^R)$ in the hybrid log-Poisson regression model (2) in loss reserving framework.
We define the fuzzy goodness of fit index by :

$$\tilde{R}_F^2 = \frac{FSSR}{FSST} = 1 - \frac{FSSE}{FSST} \qquad (15)$$

$$= \frac{||Y^{c^*} - \mathbf{1}\overline{Y}_{\ln}^c||^2 + ||Y^{L^*} - \mathbf{1}\overline{Y}_{\ln}^L||^2 + ||Y^{R^*} - \mathbf{1}\overline{Y}_{\ln}^R||^2}{||Y^{c'} - \mathbf{1}\overline{Y}_{\ln}^c||^2 + ||Y^{L'} - \mathbf{1}\overline{Y}_{\ln}^L||^2 + ||Y^{R'} - \mathbf{1}\overline{Y}_{\ln}^R||^2} \qquad (16)$$

$$= 1 - \frac{||Y^{c'} - Y^{c^*}||^2 + ||Y^{L'} - Y^{L^*}||^2 + ||Y^{R'} - Y^{R^*}||^2}{||Y^{c'} - \mathbf{1}\overline{Y}_{\ln}^c||^2 + ||Y^{L'} - \mathbf{1}\overline{Y}_{\ln}^L||^2 + ||Y^{R'} - \mathbf{1}\overline{Y}_{\ln}^R||^2} \qquad (17)$$

Algorithm for implementation

1) **Modelling of Y_{ij} with the log-Poisson model :**
One estimate the incremental losses Y_{ij} through log-Poisson regression, i.e

$$Y_{ij} \sim \mathcal{P}(e^{\nu_{ij}}), \quad \text{where } \nu_{ij} = \tau + \alpha_i + \gamma_j$$

$$\Rightarrow \hat{Y}_{ij} = e^{\mathbf{X}_i^T \hat{\beta}}, \ (i,j) \in \{1,\ldots,k\} \times \{1,\ldots,k-i+1\}$$

2) **Estimation of $Y_{ij}^R, Y_{ij}^c, Y_{ij}^L$:**
One assume that the incremental losses Y_{ij} have been modelled as follows:

$$Y_{ij} \sim \mathcal{P}(\varphi_{ij}), \ \varphi_{ij} = e^{\tau+\alpha_i+\gamma_j} \tag{18}$$

we compute now the Pearson residuals

$$\hat{r}_{p_{ij}} = \frac{Y_{ij} - \hat{\varphi}_{ij}}{\sqrt{\hat{\varphi}_{ij}}}$$

where $\hat{\varphi}_{ij} = e^{\hat{\tau}+\hat{\alpha}_i+\hat{\gamma}_j}$

and $(i,j) \in \{1,\ldots,k\} \times \{1,\ldots,k-i+1\}$

The adjusted residuals (England and Verrall, 1999) are computed as follows:

$$\hat{r}'_{p_{ij}} = \sqrt{\frac{n}{n-p}} \hat{r}_{p_{ij}},$$

$$\text{where } n = \frac{k(k+1)}{2}, \ p = 2k-1 \tag{19}$$

Algorithm for implementation

Our idea is to construct the fuzzy output $\tilde{Y}_{ij}$ as follow :

- $Y^c_{ij} = Y_{ij}$
- $Y^L_{ij} = Y_{ij} - \dfrac{|\hat{r}'_{p_{ij}}|}{2}$
- $Y^R_{ij} = Y_{ij} + \dfrac{|\hat{r}'_{p_{ij}}|}{2}$

3) **Estimation of parameters in the new model :** One can use the expression of $\hat{\beta}, \hat{\theta}, \hat{\delta}, \hat{\lambda}$ and $\hat{\mu}$ given in theorem 1 to estimate the fuzzy parameters of the new model in equation (2). For that, an iterative algorithm have been written under R to estimate those parameters.

4) **Estimation of the goodness of fit and MSEP:** From step **3)**, one can get

$$\hat{Y}^{c^*} = \mathbf{X}\hat{\beta} \quad (20)$$

$$\hat{Y}^{L^*} = \mathbf{X}\hat{\beta}\hat{\theta} + \mathbf{1}\hat{\lambda} \quad (21)$$

$$\hat{Y}^{R^*} = \mathbf{X}\hat{\beta}\hat{\delta} + \mathbf{1}\hat{\mu} \quad (22)$$

Then from definition 4, the estimation of $\tilde{R}^2_F$ is given by

$$\tilde{R}^2_F = 1 - \frac{||Y^{c'} - \hat{Y}^{c^*}||^2 + ||Y^{L'} - \hat{Y}^{L^*}||^2 + ||Y^{R'} - \hat{Y}^{R^*}||^2}{||Y^{c'} - \mathbf{1}\overline{Y}^c_{\ln}||^2 + ||Y^{L'} - \mathbf{1}\overline{Y}^L_{\ln}||^2 + ||Y^{R'} - \mathbf{1}\overline{Y}^R_{\ln}||^2} \quad (23)$$

Algorithm for implementation

The common formula to compute the Mean Square Error Prediction of incremental losses is

$$\mathbf{MSE} = \frac{1}{n}\sum_{i,j}(Y_{ij} - \hat{Y}_{ij})^2, \quad \hat{Y}_{ij} = \mathbb{E}_F(\hat{\tilde{Y}}_{ij}, \pi) \tag{24}$$

where

- π is the decision-maker risk aversion parameter $(0 \leqslant \pi \leqslant 1)$.
- $\mathbb{E}_F(\cdot)$ denotes the expected value of FN.

5) Estimation of outstandings reserves :
At this step, we predict $\tilde{Y}_{ij}$ using the new model as follows :

$$\hat{Y}_{ij}^{c\prime} = \mathbf{X}_i^T\hat{\beta} \Rightarrow \hat{Y}_{ij}^{c} = e^{\mathbf{X}_i^T\hat{\beta}} \tag{25}$$

$$\hat{Y}_{ij}^{L\prime} = \mathbf{X}_i^T\hat{\beta}\hat{\theta} + \hat{\lambda} \Rightarrow \hat{Y}_{ij}^{L} = e^{\mathbf{X}_i^T\hat{\beta}\hat{\theta}+\hat{\lambda}} \tag{26}$$

$$\hat{Y}_{ij}^{R\prime} = \mathbf{X}_i^T\hat{\beta}\hat{\delta} + \hat{\mu} \Rightarrow \hat{Y}_{ij}^{R} = e^{\mathbf{X}_i^T\hat{\beta}\hat{\delta}+\hat{\mu}} \tag{27}$$

where

$$(i,j) \in \{1,\dots,k\} \times \{k-i+1,\dots,k\}$$

$$\hat{\beta} = \left(\hat{\tau}, \hat{\alpha}_2, \dots, \hat{\alpha}_k, \hat{\gamma}_2, \dots, \hat{\gamma}_k\right)^T \in \mathbb{R}^{2k-1}$$

$$\hat{\tilde{Y}}_{ij} = (\hat{Y}_{ij}^{L}, \hat{Y}_{ij}^{c}, \hat{Y}_{ij}^{R})$$

Algorithm for implementation

Then the fuzzy total loss reserve is computed as follow:

$$\begin{aligned}\tilde{R}_{T.Res} &= \sum_{i=1}^{k} \sum_{j=k-i+1}^{k} \hat{\tilde{Y}}_{ij} \\ &= \left(\sum_{i=1}^{k} \sum_{j=k-i+1}^{k} \hat{Y}_{ij}^{L}, \sum_{i=1}^{k} \sum_{j=k-i+1}^{k} \hat{Y}_{ij}^{c}, \sum_{i=1}^{k} \sum_{j=k-i+1}^{k} \hat{Y}_{ij}^{R} \right) \\ &= \left(R_{T.Res}^{L}, R_{T.Res}^{c}, R_{T.Res}^{R} \right)\end{aligned}$$

In this article, the concept of expected value of FN (de Campos Ibáñez and Muñoz, 1989) to move from the fuzzy value of total loss reserve $\tilde{R}_{T.Res}$ to the crisp value of total loss reserve $R_{T.Res}$ has been used.

The $h-$level of fuzzy total loss reserve is defined as follow :

$$\tilde{R}_{T.Res}(h) = \left[h \cdot R_{T.Res}^{c} - (1-h) \cdot R_{T.Res}^{L}; h \cdot R_{T.Res}^{c} + (1-h) \cdot R_{T.Res}^{R} \right]$$

Then the expected value of FN $\tilde{R}_{T.Res}$ is defined as follows :

$$\mathbb{E}_F(\tilde{R}_{T.Res}, \pi) = (1-\pi) \int_0^1 \left(h \cdot R_{T.Res}^{c} - (1-h) \cdot R_{T.Res}^{L} \right) dh + \pi \int_0^1 \left(h \cdot R_{T.Res}^{c} + (1-h) \cdot R_{T.Res}^{R} \right) dh$$

Algorithm for implementation

Hence we have

$$\begin{aligned}\mathbb{E}_F(\tilde{R}_{T.Res},\pi) =&(1-\pi)\int_0^1 \Big(h\cdot R^c_{T.Res} - R^L_{T.Res} + h\cdot R^L_{T.Res}\Big)dh+\\ &\pi\int_0^1 \Big(h\cdot R^c_{T.Res} + R^R_{T.Res} - h\cdot R^R_{T.Res}\Big)dh\\ =&(1-\pi)\int_0^1 h(R^c_{T.Res}+R^L_{T.Res})dh-\\ &(1-\pi)\int_0^1 R^L_{T.Res}dh + \pi\int_0^1 h(R^c_{T.Res}-R^R_{T.Res})dh+\\ &\pi\int_0^1 R^R_{T.Res}dh\\ =&\frac{(1-\pi)(R^c_{T.Res}+R^L_{T.Res})}{2} - (1-\pi)R^L_{T.Res}+\\ &\frac{\pi(R^c_{T.Res}-R^R_{T.Res})}{2} + \pi R^R_{T.Res}\\ =&\frac{(1-\pi)(R^c_{T.Res}-R^L_{T.Res})}{2} + \frac{\pi(R^c_{T.Res}+R^R_{T.Res})}{2}\end{aligned}$$

Test on real data

- Let us apply both the classical log-Poisson regression (Mack, 1991) and the hybrid model estimated by a FLS procedure on a real data.

- One can use any *run-off triangle* as a numerical example. For example, let's use the numerical example from de Andrés Sánchez (2006) (Table 1)

	i/j	0	1	2	3	4
		Development Year				
Origin Year	2000	1120	2090	2610	2920	3130
	2001	1030	1920	2370	2710	
	2002	1090	2140	2610		
	2003	1300	2650			
	2004	1420				

Table: **1.** Numerical example from de Andrés Sánchez (2006)

- As an example of reading the *run-off triangle* (Table 1), we can say that : 1030 is the indemnity amount of an accident occurred in 2001 and paid during the same year.
- 1920 is the indemnity amount of a claim occurred in 2001 and paid in 2002.

Test on real data

- According to the $\mathbf{1^{st}}$ step of the algorithm, one perform the classical log-Poisson regression on the data from Table 1 using **R** software.
- Estimated parameters are displayed in Table 2

$(\hat{\alpha}_i)_{2\leqslant i\leqslant 5}$	$(\hat{\gamma}_j)_{2\leqslant j\leqslant 5}$	$p-$ value$(\hat{\alpha}_i)$	$p-$ value$(\hat{\gamma}_j)$
-0.084	0.661	$4.23\times e^{-08}$	$<2\times e^{-16}$
0.005	0.865	0.741	$<2\times e^{-16}$
0.207	0.987	$<2\times e^{-16}$	$<2\times e^{-16}$
0.262	1.052	$4.72\times e^{-16}$	$<2\times e^{-16}$
$\hat{\tau}=6.996$		$p-$ value$(\hat{\tau})=<2\times e^{-16}$	
$\mathbf{R^2}=0.9621$	$\mathbf{MSEP}=12699$	**Total Reserve** $=33634.89$	

Table: **2.** Estimated parameters

- From Table 2 and with a threshold of 1%, one conclude that except $\hat{\alpha}_3$, the others coefficients are statistically significant.
- As an example of interpretation of the results, the estimation of the payment of 2^{nd} origin year could be $e^{6.996-0.084+0.661}=1945.9002.$

Test on real data

- The goodness of fit of the model to the data is good, since $\mathbf{R^2 = 96.21\%}$, the value of the Mean Square Error Prediction is $\mathbf{12699}$.
- We shall compare these two values ($\mathbf{R^2}$ and $\mathbf{MSEP}$) with the ones we will get from the hybrid model. The estimation of the outstanding reserves is $\mathbf{33634.89}$
- Let's perform the steps **2),3)** and **4)** of the algorithm. The iterative algorithm to estimate parameters $\beta, \theta, \delta, \lambda, \mu$ converges after 12112.

$$\hat{\boldsymbol{\beta}} = \begin{pmatrix} 7.003 \\ -0.084 \\ 0.658 \\ 0.0005 \\ 0.859 \\ 0.193 \\ 0.981 \\ 0.255 \\ 1.045 \end{pmatrix}; \ \hat{\theta} = 1.0004; \quad \hat{\lambda} = -0.003; \quad \hat{\delta} = 0.999; \qquad (28)$$

$$\hat{\mu} = 0.003; \ \tilde{R}_F^2 = 0.998; \ \mathbf{MSEP_F} = 699.66$$

Test on real data

- From the goodness of fit index, we conclude that the hybrid model is more adequate to the classical one since $\tilde{R}_F^2 > R^2$.

- From the Mean Square Error Prediction, our model predict incremental losses better than the classical log-Poisson model because $\mathbf{MSEP_F} < \mathbf{MSEP}$.

- From step **5)** of our algorithm, we can predict the incremental losses as FN and total fuzzy loss reserve $\tilde{\mathbf{R}}_{\mathbf{T.Res}} = (33384.915, 33386.738, 33388.281)$

- Using the expected value of FN for defuzzification purposes, one can compute the crisp value of outstanding loss reserves with the maximum decision-maker risk aversion, i.e $\pi = 1$.

$$\mathbb{E}_F(\tilde{R}_{T.Res}, \pi = 1) = \hat{R}_{T.Res} = \frac{R^c_{T.Res} + R^R_{T.Res}}{2} = \mathbf{33387.5095}$$

- From the comparison of the goodness of fit indices and MSEP, one conclude that the hybrid model fits well with the data and gives better predictions compared to the classical model. Therefore the outstanding reserves have been optimized since our model approaches the fair value of reserves better than the classical log-Poisson model.

Conclusion

- This paper has considered the relevance of hybrid models in loss reserving framework, mainly when we are in presence of vague information (Straub and Swiss, 1988).
- The parameters of a hybrid log-Poisson model for loss reserving purposes has been estimated using the FLS procedures.
- Furthermore we have developed a goodness of fit index to assess and compare our model with the classical log-Poisson model.
- According to the goodness of fit and the MSEP, the hybrid model approaches the fair value of loss reserves better than the well known log-Poisson regression model.
- However, since we got an iterative estimator, the **R** program take some time to converge (**12112** iterations). This is due to the estimation steps which are based on a sequence of interpolations.

Thank you for your attention !