Impact of an aggregation tree over the skewness of the final risk distribution

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ABSTRACT

Following the implementation of Solvency 2 in Europe and other parts of the world, many (re)insurance companies decided to put in place Internal Models. The central part of an internal model is its aggregation tree and the calibration of the aggregation tree has been the subject of many articles. Among the different articles, "Estimating copula for insurance from scarce observations, expert opinion and prior information: a Bayesian approach", ASTIN Bulletin (2012) by Arbenz et al. sets a general framework for estimating dependencies.

In practice, most of the calibration exercises focus on elements which are similar in nature to "correlation" as this is the easiest part of the dependence structure: Practitioners (including underwriters, claims managers, actuaries and finance managers) usually understand and have a slight feeling for correlations if relevant explanations are provided.

But the other impacts of these calibration choices are usually forgotten as practitioners do not have a view on these elements. In particular, the number of layers of the aggregation tree as well as the order of the aggregation play a role on both the overall volatility and the overall skewness of the final risk distribution. But, usually, these elements are not discussed in detail. As volatility and skewness are the main determinants of the risk measure (Value at Risk or Expected Shortfall), understanding the impact of the calibration choices on these elements is certainly important and should also be in the focus of the calibration exercise.

As a consequence, this article aims at estimating the impact of the different choices of calibration of the aggregation tree on both volatility and skewness of the overall risk distribution. Examples of calibration are provided to demonstrate the impact of the different (forgotten) elements of choice in a calibration exercise.

KEYWORDS

Coefficient of variation, diversification, value at risk, skewness, kurtosis, aggregation tree, copula, Cornish-fisher expansion, Kendall Tau.

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1. INTRODUCTION

When a (re)insurance company develops an internal model, it is faced with the question of the aggregation of its risks. The usual answer to this question is the development of an aggregation tree with different layers of risk aggregation. Once the aggregation tree is fixed, the calibration of the dependence at each node of the aggregation tree is the next challenge. Such challenge is usually resolved by mixing expert opinions with some prior knowledge of the dependence (see "Estimating copula for insurance from scarce observations, expert opinion and prior information: a Bayesian approach", ASTIN Bulletin (2012) by Arbenz et al.). Therefore, in practice, most of the calibration exercise focuses on elements which are similar in nature to "correlation" as this is the easiest part of the dependence structure: Practitioners (including underwriters, claims managers, actuaries and finance managers) usually have a slight feeling for correlations if relevant explanations are provided.

In view of the simple nature of the correlation-type aggregation, it is very often the case that internal models will be using advanced dependency structures, in particular, copula structures. However, many types of copula structures exist and it is not always easy to decide which one to use. As for this article, the following 2 main groups of copulas are considered (see Embrechts, Lindskog, McNeil (2001)): The Elliptical group (comprising in particular the Gaussian copula and the t-Student copula) and the Archimedean group (comprising in particular the Clayton copula). These 2 groups include most of the copulas used in practice by the industry. Within each group, each copula has its own properties which can fit some specific needs in relation to the objectives of the aggregation.

Following the choice of the copula and after calibration of the aggregation nodes, the other impacts of the calibration choice are usually forgotten, in particular, the resulting skewness of the final risk distribution. This is mostly due to the fact that practitioners do not have any benchmarks related to the higher moments of the risk distribution. Even though the measures of multivariate skewness and kurtosis have been studied by Koll (2008), Mardia (1970) and Schwager (1983), such elements are therefore not taken into account in calibration exercises.

The aim of this article is to provide a few simple examples of skewness estimation and to see on a simple aggregation tree the impact of the change of the aggregation scheme on the skewness and hence, on the capital requirements.

Remark: The R code developed to estimate the presented formulae is available on the URL:
https://drive.google.com/open?id=1tX324aD6HqQI1ZA8ghVcOyR2SV-ldH9A

2. SKEWESS OF TWO AGGREGATED RANDOM VARIABLES

As mentioned in the introduction, this chapter will focus on the aggregation of two random variables with a focus on the skewness of the aggregated distribution.

a. Theoretical skewness estimation

Let's assume two random variables X_1 and X_2 for which we know their joint copula distribution. For this first section, we will focus on the third moment of their aggregated distribution i.e. on

$$SK(X_1 + X_2) = E\left[\left((X_1 + X_2) - E(X_1 + X_2)\right)^3\right]$$

where we define the third moment of X_1 as:

$$SK(X_1) = E\left[\left(X_1 - E(X_1)\right)^3\right] = E(X_1^3) - 3E(X_1)E(X_1^2) + 2E(X_1)^3$$

It has to be noted that the joint distribution can be reflected through any copula. For this article, we will limit our numerical examples to the Gaussian, Gumbel and mirrored Clayton copula. These copulas cover some of the most used copulas in the industry.

We have the following:

$$E\left[\left((X_1 + X_2) - E(X_1 + X_2)\right)^3\right] = E[(X_1 + X_2)^3] - 3E[(X_1 + X_2)^2]E[X_1 + X_2] + 2E[X_1 + X_2]^3$$
$$= E(X_1^3) + 3E(X_1^2X_2) + 3E(X_1X_2^2) + E(X_2^3)$$
$$-3E(X_1)E(X_1^2) - 6E(X_1)E(X_1X_2) - 3E(X_1)E(X_2^2) - 3E(X_2)E(X_1^2) - 6E(X_2)E(X_1X_2) - 3E(X_2)E(X_2^2)$$
$$+2E(X_1)^3 + 6E(X_1)^2E(X_2) + 6E(X_2)^2E(X_1) + 2E(X_2)^3$$

Therefore:

$$SK(X_1 + X_2) = SK(X_1) + SK(X_2) + 3Cov(X_1^2, X_2) + 3Cov(X_1, X_2^2) - 6[E(X_1) + E(X_2)]Cov(X_1, X_2)$$

We will now use the following general property of copulas:
If $C_{X,Y}$ is the copula of (X,Y), any increasing transformation of (X,Y) has the same copula i.e.

- If f is increasing and g is increasing, $C_{f(X),g(Y)}(u,v) = C_{X,Y}(u,v)$

With this property, we know that the dependence between X_1 and X_2 and X_1^2 and X_2 is the same i.e.:

$$Cov(X_1, X_2) = \rho\sigma_{X_1}\sigma_{X_2}$$
$$Cov(X_1^2, X_2) = \rho\sigma_{X_1^2}\sigma_{X_2}$$

As a result, we are interested in the finding the parameter α_1 of the following equation:

$$Cov(X_1^2, X_2) = \alpha_1\, Cov(X_1, X_2)$$
$$\alpha_1 = \frac{\sigma_{X_1^2}}{\sigma_{X_1}}$$

If we denote:

$$KT(X) = E\left[\left(X - E(X)\right)^4\right]$$
$$SK(X) = E\left[\left(X - E(X)\right)^3\right]$$
$$Var(X) = E\left[\left(X - E(X)\right)^2\right]$$

we have:

$$Var(X^2) = E\left[\left(X^2 - E(X^2)\right)^2\right]$$
$$Var(X^2) = KT(X) + 4SK(X)\,E(X) + 4\,E(X)^2\,Var(X) - Var(X)^2$$

Hence:

$$\alpha_1 = \frac{\sigma_{X_1^2}}{\sigma_{X_1}} = \sqrt{\frac{KT(X_1) + 4SK(X_1)\,E(X_1) + 4\,E(X_1)^2\,Var(X_1) - Var(X_1)^2}{Var(X_1)}}$$

And:

$$SK(X_1 + X_2) = SK(X_1) + SK(X_2) + 3[\alpha_1 + \alpha_2 - 2E(X_1) - 2E(X_2)]Cov(X_1, X_2) \quad (1)$$

There remains the issue of estimating the coefficient of correlation for any case. In the case of elliptical copulas, this coefficient is given directly by the correlation matrix. However, in the case of an archimedian copula, such coefficient is not given directly and a transformation needs to be done on the basis of the Kendall Tau. Referring to Embrechts et al. (2015), we will use the transformation below.

Let's assume we know the Kendall Tau τ of our dependence structure, Embrechts et al. (2015) propose the following transformation:

$$\rho = \sin\left(\frac{\pi}{2}\tau\right)$$

From the above, it is possible to estimate the Spearman Rho:

$$\rho_S = \frac{6}{\pi}\arcsin\left(\frac{\rho}{2}\right)$$

In the following, we will use the Spearman Rho as a proxy for the correlation coefficient in equation (1) in the case of archimedian copulae. For the Gaussian copula, we will just be using the standard correlation ρ. Limitations related to such a proy are going to be tested in the next paragraph.

b. Numerical applications

The formula (1) was tested on different dependence structure and different marginal distributions. The tests consisted in simulating 100 000 random draws from the distributions and to aggregate these simulations on using the proposed dependence structure. The simulations were produced in R using RStudio. The resulting skewness of the aggregated distribution is then compared to the proxy provided by formula (1).

Gaussian copula[2]

	Param. 1	Param. 2	Correlation	0.174	0.309	0.415	0.5	0.568	0.623	0.669	0.707	0.997
Normal	20.000	20.000	Simulations	-0.004	-0.006	0.001	-0.006	0.003	-0.003	0.001	0.000	-0.002
	10.000	10.000	Proxy	-0.004	-0.001	-0.001	-0.004	0.002	-0.001	-0.004	-0.001	-0.002
LogNormal	2.991	0.100	Simulations	0.927	0.931	0.917	0.924	0.938	0.931	0.931	0.942	0.932
	2.228	0.385	Proxy	0.929	0.934	0.913	0.928	0.936	0.929	0.933	0.940	0.932
Pareto	8.540	25.860	Simulations	3.155	2.968	3.104	3.068	3.270	3.191	3.158	3.080	3.273
	5.770	32.910	Proxy	3.155	2.981	3.110	3.058	3.279	3.183	3.158	3.079	3.285
Weibull	8.933	30.940	Simulations	-0.273	-0.275	-0.271	-0.272	-0.273	-0.264	-0.269	-0.275	-0.272
	5.381	43.171	Proxy	-0.273	-0.275	-0.272	-0.270	-0.271	-0.268	-0.271	-0.271	-0.270

Table 1: Skewness comparison between simulated distribution and formula (1 – Proxy) for gaussian copula

Overall, in the case of the Gaussian copula, the proposed proxy provides a very good alternative to the simulations.

[2] Parameter 1 correspond to μ for normal and lognormal distributions, α for Pareto and k for Weibull, Parameter 2 correspond to σ for normal and lognormal distributions, λ for Pareto and Weibull.

Mirrored Clayton copula

	Param. 1	Param. 2	Theta	0.25	0.5	0.75	1	1.25	1.5	1.75	2	40
Normal	20.000	20.000	Simulations	0.642	0.721	0.748	0.762	0.763	0.756	0.743	0.732	0.585
	10.000	10.000	Proxy	0.566	0.619	0.645	0.669	0.687	0.696	0.700	0.703	0.731
LogNormal	2.991	0.100	Simulations	1.108	1.200	1.214	1.213	1.199	1.204	1.183	1.160	0.930
	2.228	0.385	Proxy	1.017	1.067	1.085	1.102	1.104	1.141	1.138	1.130	1.131
Pareto	8.540	25.860	Simulations	3.474	3.619	3.827	3.990	3.711	3.940	3.757	3.669	3.478
	5.770	32.910	Proxy	3.622	3.820	4.175	4.570	4.112	4.701	4.434	4.296	4.376
Weibull	8.933	30.940	Simulations	-0.148	-0.092	-0.065	-0.056	-0.061	-0.065	-0.072	-0.082	-0.355
	5.381	43.171	Proxy	-0.288	-0.299	-0.306	-0.307	-0.310	-0.309	-0.310	-0.312	-0.317

Table 2: Skewness comparison between simulated distribution and formula (1 – Proxy) for M-Clayton copula

In the case of the mirrored Clayton copula, the proposed proxy provides a good alternative to the simulations for the normal, lognormal and Pareto cases. For Weibull, due to the negative skewness of the distribution, the proxy does not seem to work.

Gumbel copula

	Param. 1	Param. 2	Theta	1.125	1.25	1.375	1.5	1.625	1.75	1.875	2	21
LogNormal	2.991	0.100	Simulations	1.089	1.121	1.146	1.110	1.114	1.097	1.078	1.072	0.928
	2.228	0.385	Proxy	1.028	1.060	1.112	1.096	1.126	1.135	1.129	1.140	1.145
Pareto	8.540	25.860	Simulations	3.540	3.575	3.760	3.662	3.736	3.831	3.994	3.751	3.567
	5.770	32.910	Proxy	3.623	3.748	4.248	4.147	4.345	4.547	5.069	4.541	4.624

Table 2: Skewness comparison between simulated distribution and formula (1 – Proxy) for M-Clayton copula

In the case of the Gumbel copula, the proposed proxy provides a good alternative to the simulations for the lognormal and Pareto cases.

Overall, in the tested cases, the formula (1) seems therefore applicable to estimate the impact of an aggregation of 2 risk distributions over the resulting skewness. We will therefore use this formula in the next sections of this article.

c. Limitations and comments

From equation (1), we can see that the aggregated skewness is made of two components:

- As expected, the first component is made of the sum of the skewness of the aggregated distributions:

$$SK(X_1) + SK(X_2)$$

- And the second component which is always positive:

$$3[\alpha_1 + \alpha_2 - 2E(X_1) - 2E(X_2)]Cov(X_1, X_2)$$

  as: $\alpha_1 \geq 2E(X_1)$

  In the following, we will denote f as:

$$f = [\alpha_1 + \alpha_2 - 2E(X_1) - 2E(X_2)]$$

Therefore, when aggregating two random variable, we will have the following impacts:

- A reduction of the overall Coefficient of Variation due to the diversification between risks when the correlation matrix is positive definite (see below for more comments on this);
- An increase in skewness due to equation (1).

The next section will focus on the comparison between diversification and increase in skewness to decide which of the two elements is more important.

As for the diversification effect, in Embrechts et al. (2015), the following limitations are mentioned when using Kendall Tau to calibrate a copula: Suppose we wish to estimate the correlation matrix using the equation:

$$\tau(X_i, X_j) = \left(\frac{2}{\pi}\right) arcsin(\rho_{ij})$$

There is no guarantee that the resulting matrix of rank correlation coefficients will remain positive definite. However, it is possible to find the closest positive definite correlation matrix using Higham (2002) algorithm.

3. IMPACT OF SKEWNESS ON CAPITAL REQUIREMENTS

In order to estimate the impact of the skewness on capital requirements, we will rely on the Cornish-Fisher expansion. This expansion allows the estimation of any quantile of a distribution based on the knowledge of its first 3 moments. Even though such estimation is not very precise, it will be a way to see how much the impact of the aggregation tree on the skewness influences the capital requirements.

This section will therefore introduce, in a first part, the Cornish-Fisher expansion and, in a second part, try to compare the impact of the diversification benefit resulting from any aggregation against the increase of skewness resulting from the same aggregation. Finally, we will apply these results to two aggregation trees which initial leaf-nodes are the same: Only the successive aggregation steps will differ. As a result, it will be possible to estimate the impact of the aggregation tree on the skewness of the aggregated distribution and, as a consequence, estimate the difference in capital requirements.

a. Cornish-Fisher expansion

In 1938, the Cornish-Fisher expansion (see Cornish et al 1938) introduced an approximation for the quantiles of a random variable based on its first few cumulants. For this expansion, let X be a random variable with density function f(x) with mean 0 and variance 1. Let β_1 be the skewness of this distribution. Let Z be a normally distributed random variable and let z_α be the αth quantile of this distribution. Then the α^{th} quantile ω_α of the distribution X can be approximated by:

$$\omega_\alpha = z_\alpha + \frac{1}{6}(z_\alpha^2 - 1)\beta_1$$

When using the Cornish-Fisher expansion for a random variable with mean R and variance σ, the quantile Q_α is estimated as:

$$Q_\alpha = R + \sigma\left(z_\alpha + \frac{1}{6}(z_\alpha^2 - 1)\beta_1\right) \qquad (2)$$

In the next section, we will use equation (2) to compare the impact on capital requirements due to diversification benefit against the skewness increase.

b. Comparison of diversification benefit against skewness increase

Let's denote: ${}_xVaR_\alpha$ the value at risk above the mean of the reserve at the quantile α. From equation (2), we have:

$${}_xVaR_\alpha = \sigma\left(z_\alpha + \frac{1}{6}(z_\alpha^2 - 1)\beta_1\right)$$

Now, when aggregating two random variables X_1 and X_2, which correlation is ρ, we have:

- Variance: $Var(X_1 + X_2) = Var(X_1) + Var(X_2) + 2\rho\sqrt{Var(X_1)Var(X_2)}$

  Let's denote D the diversification benefit which can be defined here as:

  $$Var(X_1 + X_2) = (1 - D)^2\left(\sqrt{Var(X_1)} + \sqrt{Var(X_2)}\right)^2 \qquad (3)$$

  For simplicity, let's also denote: $\sigma_{X_1} = \sqrt{Var(X_1)}$

- Skewness according to equation (1).

Using equations (1) and (3), we have:

$$_xVaR_\alpha = \sigma_{X_1+X_2}\left(z_\alpha + \frac{1}{6}(z_\alpha^2-1)\frac{SK(X_1+X_2)}{\left(\sigma_{X_1+X_2}\right)^3}\right) = z_\alpha\,\sigma_{X_1+X_2} + \frac{1}{6}(z_\alpha^2-1)\frac{SK(X_1+X_2)}{\left(\sigma_{X_1+X_2}\right)^2}$$

$$_xVaR_\alpha = z_\alpha(1-D)\ \left(\sigma_{X_1}+\sigma_{X_2}\right) + \frac{1}{6}(z_\alpha^2-1)\frac{SK(X_1)+SK(X_2) + 3\,f\,\rho\,\sigma_{X_1}\sigma_{X_2}}{\left[(1-D)\ \left(\sigma_{X_1}+\sigma_{X_2}\right)\right]^2}$$

The reduction of capital requirement due to the diversification is compared against the increase of capital requirement due to the increased skewness and summarized below:

$$D\,z_\alpha\left(\sigma_{X_1}+\sigma_{X_2}\right) \quad \text{against}\ \frac{1}{6}(z_\alpha^2-1)\frac{3\,f\,\rho\,\sigma_{X_1}\sigma_{X_2}}{(1-D)^2\left(\sigma_{X_1}+\sigma_{X_2}\right)^2}$$

In order to make this comparison, we calculate:

$$\Delta = \frac{1}{6}\frac{(z_\alpha^2-1)}{D\,z_\alpha}\frac{3\,f\,\rho\,\sigma_{X_1}\sigma_{X_2}}{(1-D)^2\left(\sigma_{X_1}+\sigma_{X_2}\right)^3} = \frac{1}{4}\frac{(z_\alpha^2-1)f}{D\,z_\alpha\left(\sigma_{X_1}+\sigma_{X_2}\right)}\left(1-\frac{\sigma_{X_1}^2+\sigma_{X_2}^2}{\left(\sigma_{X_1}+\sigma_{X_2}\right)^2}\right)$$

as $\rho\,\sigma_{X_1}\sigma_{X_2} = \frac{1}{2}\left[(1-D)^2\left(\sigma_{X_1}+\sigma_{X_2}\right)^2 - \sigma_{X_1}^2 - \sigma_{X_2}^2\right]$

Due to the complexity involved by the equations above, we will consider the following simplifications to get a first view on the value of Δ:

- We estimate Δ for the 99.5% quantile: This is consistent with the risk measure for the capital requirement. At this quantile, we have: $z_\alpha = 2.5758$
- As a simplification, we will assume that $\sigma_{X_1} = \sigma_{X_2}$.
- And we will assume that $KT(X) = 3\sigma_X^4$ and that the skewness of X is nil: This corresponds to a normal distribution. Generally, a lognormal distribution is assumed; if the coefficient of variation of a lognormal distribution is low (i.e. below 10%), then the lognormal distribution is close to a normal distribution which allows the proposed simplification.
- Which leads to $\alpha = \frac{1}{\sigma_X}\sqrt{3\sigma_X^4 + 4E(X)^2\sigma_X^2 - \sigma_X^4} = \sqrt{2\sigma_X^2 + 4E(X)^2}$
- And we need to estimate: $f = [\alpha_1 + \alpha_2 - 2E(X_1) - 2E(X_2)]$

In order to estimate a proxy for f, we calculate:

$$\frac{\alpha}{2E(X)} = \frac{\sqrt{2\sigma_X^2 + 4E(X)^2}}{2E(X)} = \sqrt{1 + \frac{\sigma_X^2}{2E(X)^2}}$$

If we denote the coefficient of variation: $CoV = \frac{\sigma_X}{E(X)}$, we know that, for usual distributions, the CoV is smaller than 30%, then we have:

$$\frac{\alpha}{2E(X)} \approx 1 + \frac{CoV_X^2}{4}$$

Hence:

$$\alpha - 2E(X) \approx E(X)\frac{CoV_X^2}{2} = \frac{\sigma_X^2}{2E(X)}$$

Therefore:

$$\Delta = \frac{1}{4}\frac{(2.5758^2-1)}{D\ 2.5758}\frac{\sigma_X^2}{2\sigma_X E(X)}\left(1-\frac{1}{2}\right) = \frac{2.1876}{16\,D}CoV_X$$

In such a case, as long as D, the diversification benefit, is not smaller than $\frac{2.1876}{16}CoV_X$ (in the case where the CoV is 30%, it would roughly be 4%), Δ is smaller than 1 which means that the diversification benefit is bigger than the skewness impact. With this simplified calculation, we can see that the skewness impact on the capital requirements is bigger than the diversification benefit only in rare cases. Such result is in conformity with the expectations.

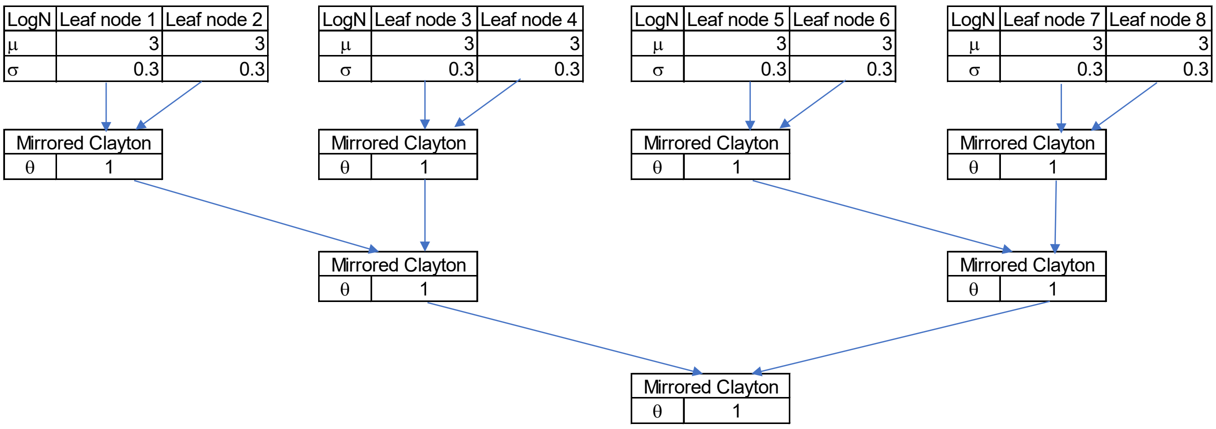
c. <u>Estimation of the impact on an example aggregation tree</u>

The last section of this article will focus on two examples with different aggregation trees.

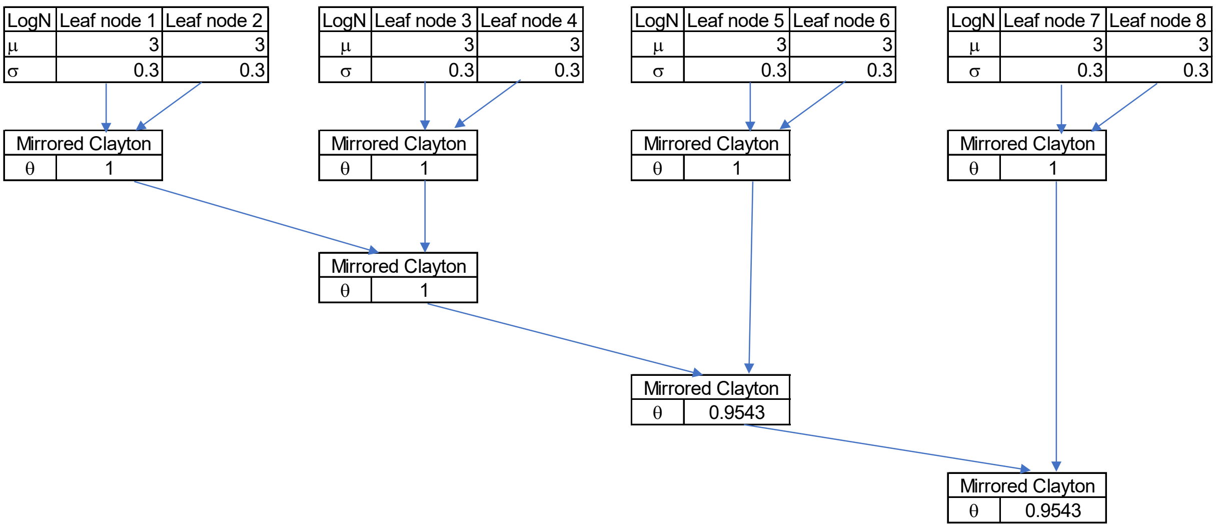
Example 1

The first example will focus on two aggregation trees: Both trees will give the same standard deviation and expected value for the aggregated distribution but they will have different intermediary aggregation layers. As a consequence, it will be possible to compare the skewness of the two resulting distributions and confirm the proxy which was presented in formula (1).

The first aggregation tree is shown below:



The second aggregation tree is shown below:

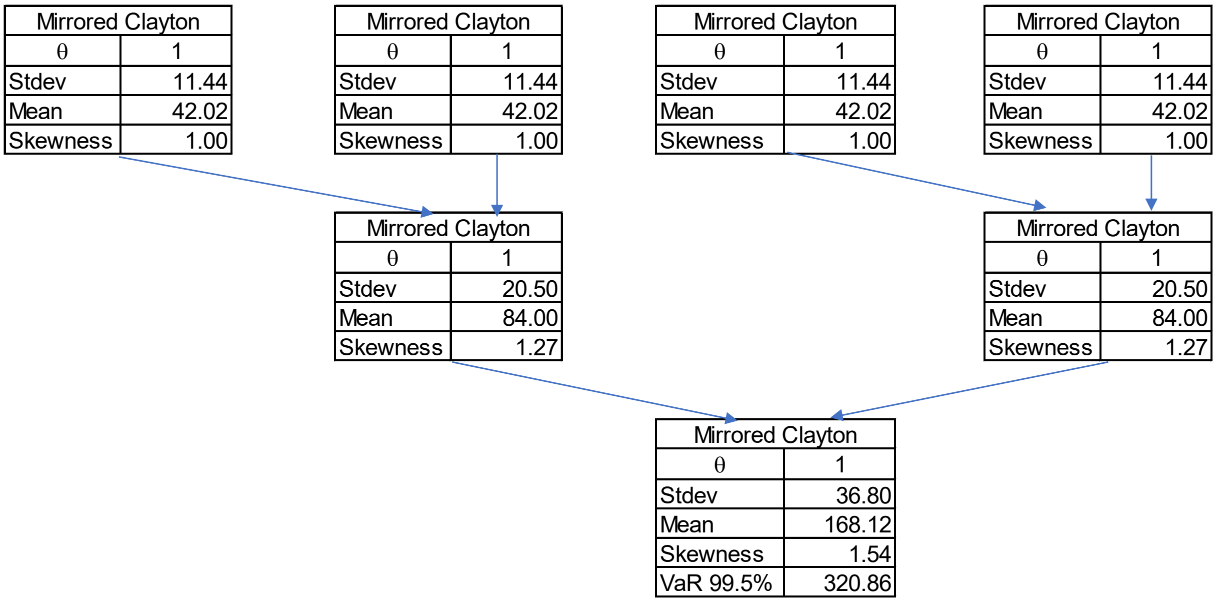


Both trees are very similar but:

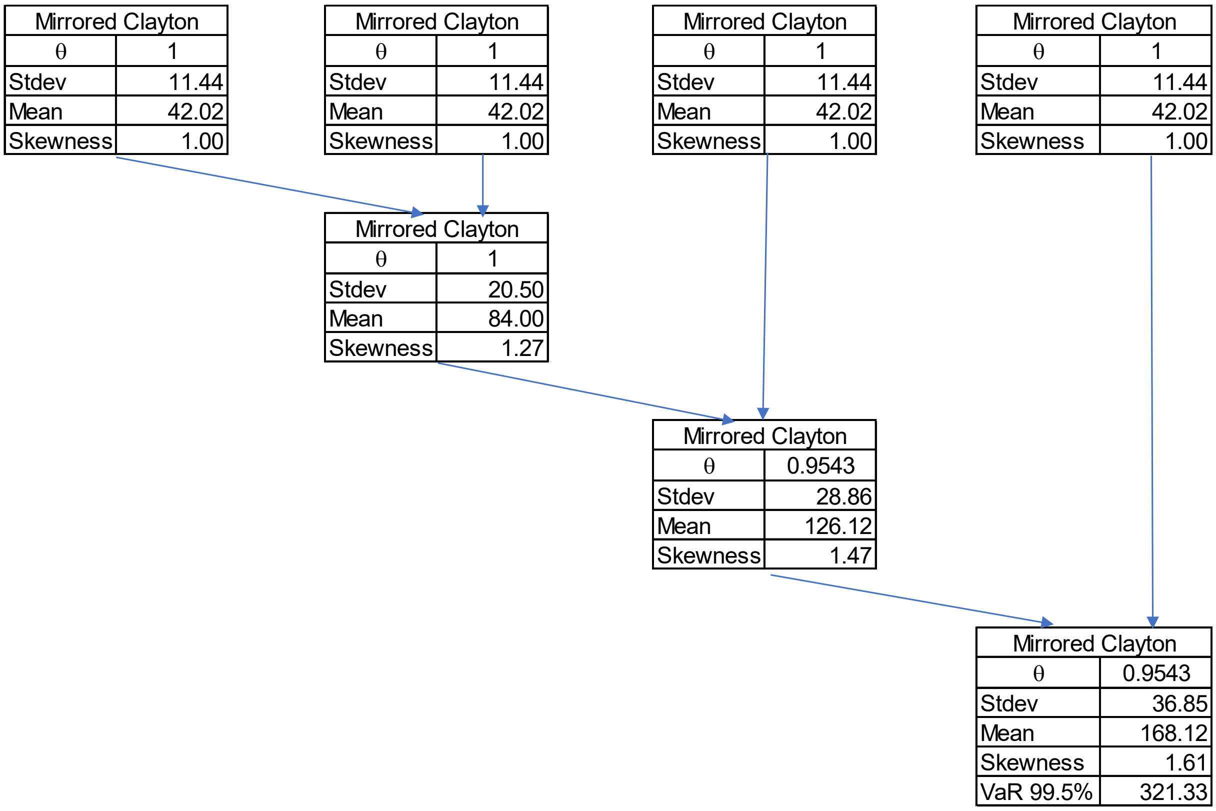
- The second tree has one more aggregation layer;
- The parameters of the last aggregation layers of the second tree are chosen so that the overall standard deviation is the same for the aggregated distributions resulting from the first and from the second trees.

The resulting mean, standard deviations and skewness at each aggregation node are shown below:

- First aggregation tree:



- Second aggregation tree:



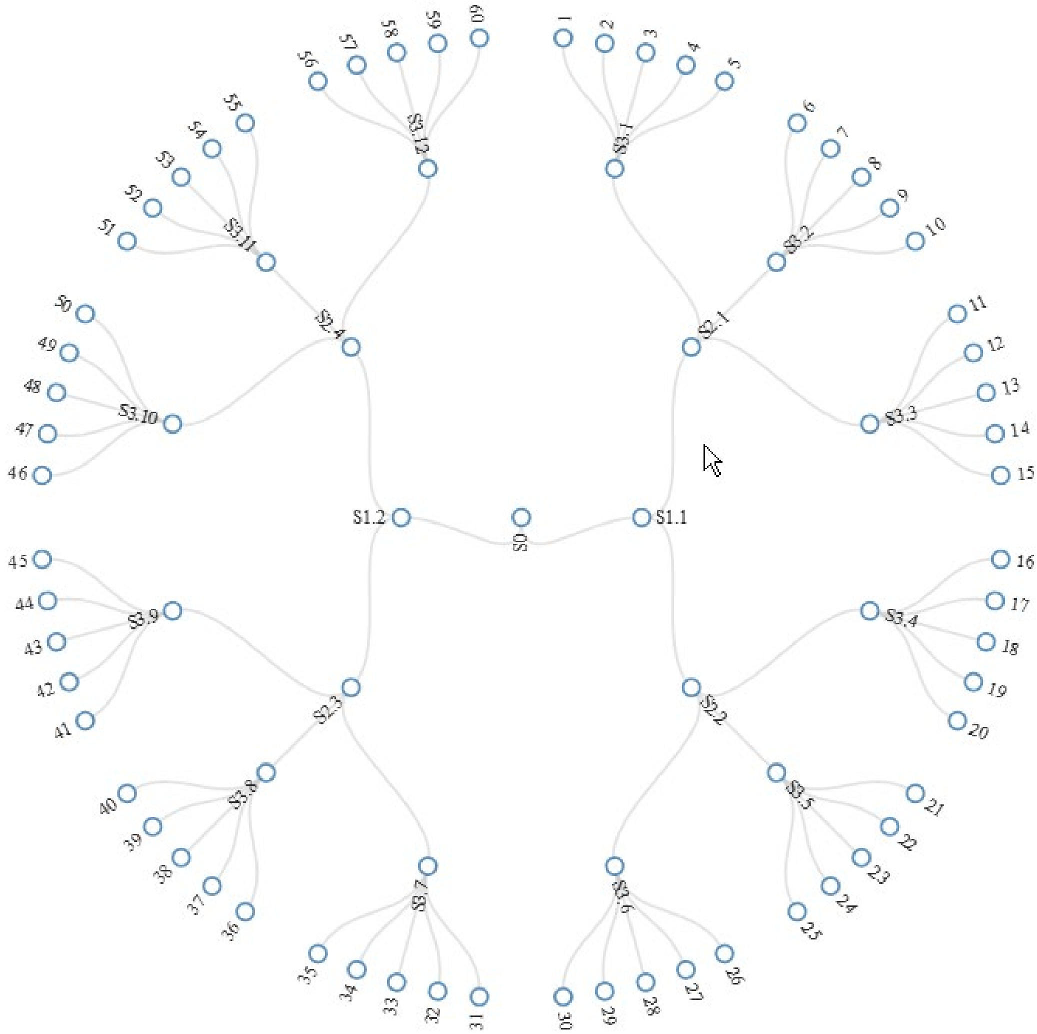
As expected, the overall skewness is higher in the second tree as compared to the first tree. This confirms the results of equation (1) in the sense that the higher the number of layers is, the higher the skewness should be (due to the second term of equation (1)). As a next step, in order to demonstrate the impact of the design of the aggregation tree, we will focus on an example where many different tree structures are tested on the same similar underlying risks

Example 2:
In this example, 60 lognormal risks are aggregated with different aggregation trees. The different aggregation trees have different layers and, for each layer, the number of aggregated risks vary. The 60 lognormal risks are the same with parameters μ equal to 3.34082 and σ equal to 0.19804. For the different trees, the same mirrored Clayton copula is used. The same theta parameter for the copula is used for all the aggregation nodes. However, different values of this parameter were tested as shown below (from very low dependence – low thetas – to high dependence – high thetas).

The different tested aggregation trees are detailed below:

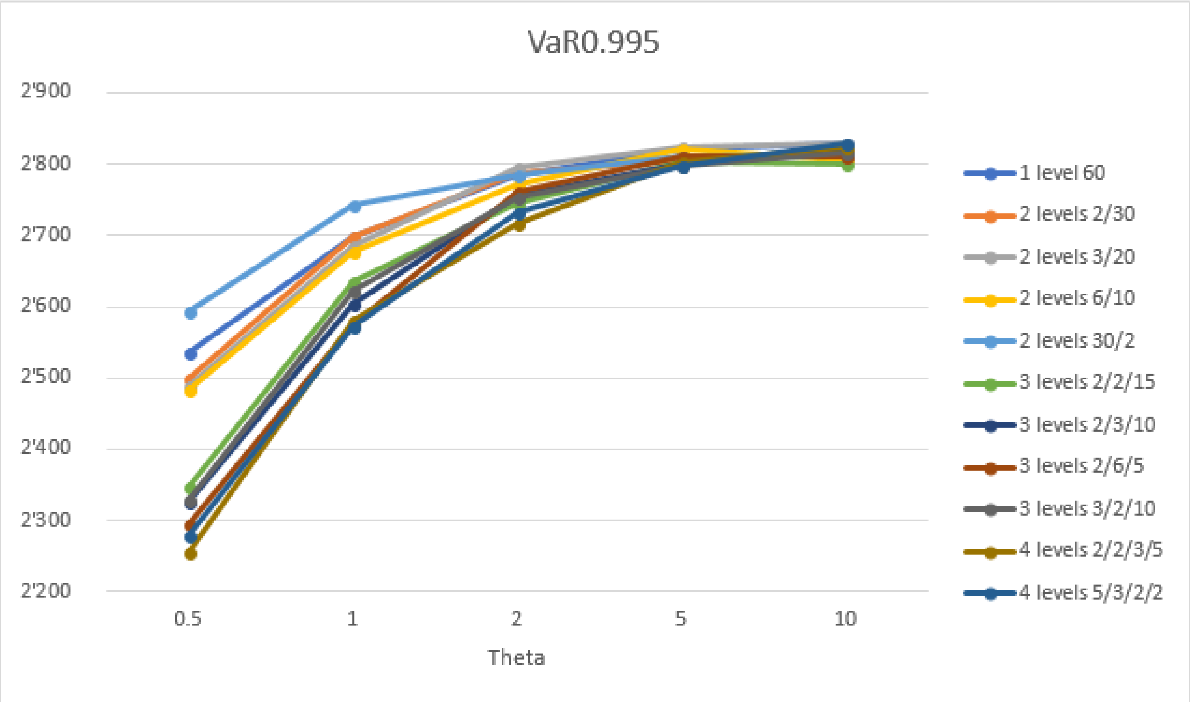
- One layer: All risks aggregated in one step
- Two layers
  - 30 risks aggregated in a first step followed by the aggregation of the resulting 2 nodes. Such aggregation will be denoted: 2 levels 2/30
  - 20 risks aggregated in a first step followed by the aggregation of the resulting 3 nodes. Such aggregation will be denoted: 2 levels 3/20
  - 10 risks aggregated in a first step followed by the aggregation of the resulting 6 nodes. Such aggregation will be denoted: 2 levels 6/10
  - 2 risks aggregated in a first step followed by the aggregation of the resulting 30 nodes. Such aggregation will be denoted: 2 levels 30/2
- Three layers
  - 15 risks aggregated in a first step followed by the aggregation of 2 nodes of 15 risks followed by the aggregation of the resulting 2 nodes. 3 levels 2/2/15
  - 10 risks aggregated in a first step followed by the aggregation of 3 nodes of 10 risks followed by the aggregation of the resulting 2 nodes. 3 levels 2/3/10
  - 5 risks aggregated in a first step followed by the aggregation of 6 nodes of 5 risks followed by the aggregation of the resulting 2 nodes. 3 levels 2/6/5
  - 10 risks aggregated in a first step followed by the aggregation of 2 nodes of 10 risks followed by the aggregation of the resulting 3 nodes. 3 levels 3/2/10
- Four layers
  - 5 risks aggregated in a first step followed by the aggregation of 3 nodes of 5 risks followed by the aggregation of the 2 nodes and followed by the aggregation of 2 nodes. 4 levels 2/2/3/5
  - 2 risks aggregated in a first step followed by the aggregation of 2 nodes of 2 risks followed by the aggregation of the 3 nodes and followed by the aggregation of 5 nodes. 4 levels 5/3/2/2

Below is an illustration of the aggregation 4 levels 2/2/3/5:

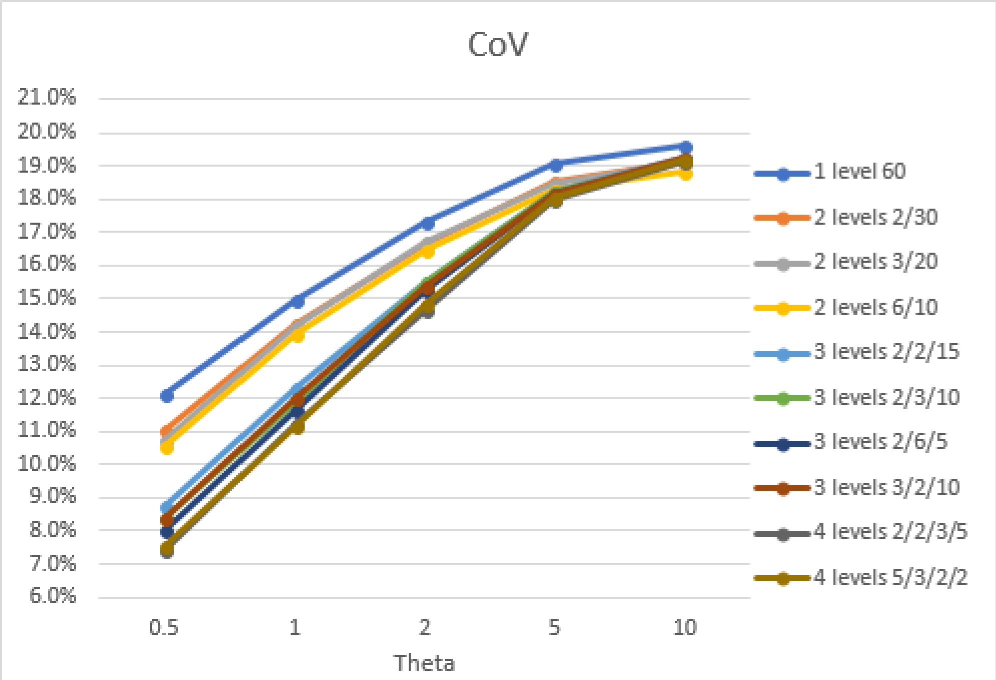


Below are a few statistics (Value at Risk 99.5%, Coefficient of Variation (CoV) and Skewness) related to the different aggregation trees:

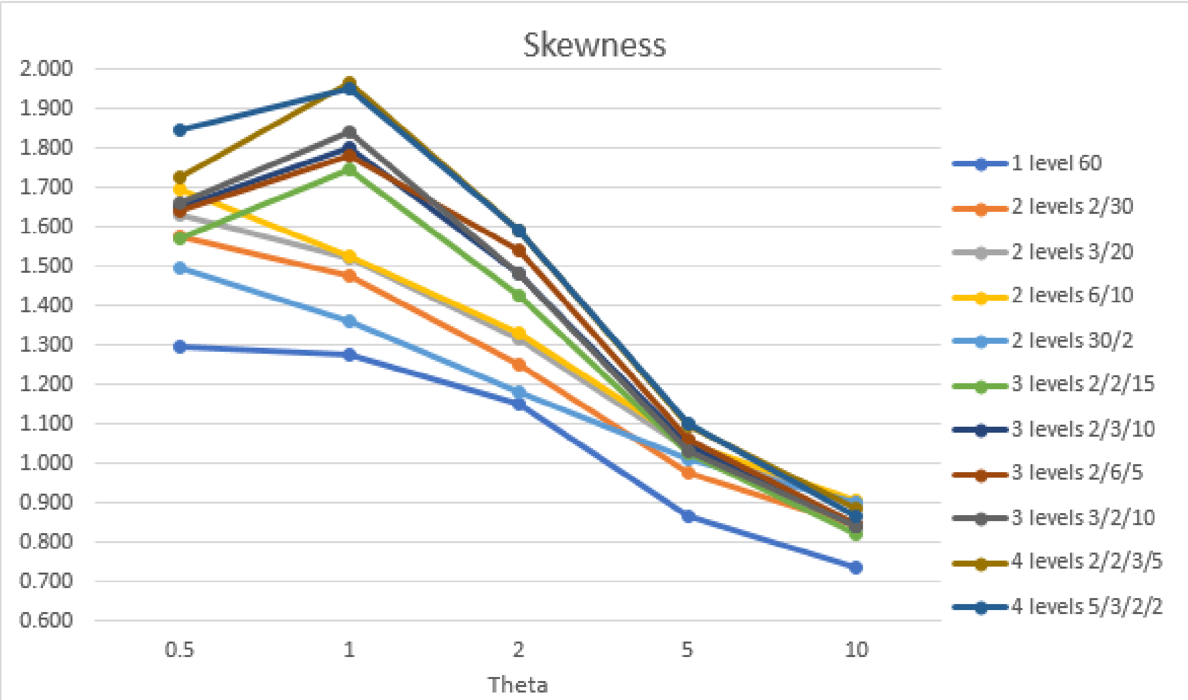
	Theta				
VaR0.995	0.5	1	2	5	10
1 level 60	2'537	2'700	2'786	2'820	2'828
2 levels 2/30	2'498	2'698	2'787	2'806	2'813
2 levels 3/20	2'489	2'684	2'795	2'825	2'829
2 levels 6/10	2'483	2'677	2'771	2'820	2'806
2 levels 30/2	2'594	2'744	2'784	2'812	2'813
3 levels 2/2/15	2'347	2'635	2'747	2'803	2'802
3 levels 2/3/10	2'327	2'604	2'757	2'801	2'819
3 levels 2/6/5	2'295	2'575	2'760	2'811	2'812
3 levels 3/2/10	2'329	2'622	2'753	2'798	2'816
4 levels 2/2/3/5	2'256	2'581	2'717	2'802	2'823
4 levels 5/3/2/2	2'280	2'572	2'733	2'798	2'829



CoV	Theta				
	0.5	1	2	5	10
1 level 60	12.1%	14.9%	17.3%	19.0%	19.6%
2 levels 2/30	11.0%	14.3%	16.7%	18.5%	19.1%
2 levels 3/20	10.7%	14.2%	16.7%	18.4%	19.1%
2 levels 6/10	10.6%	13.9%	16.5%	18.3%	18.8%
3 levels 2/2/15	8.7%	12.3%	15.5%	18.2%	19.2%
3 levels 2/3/10	8.4%	11.9%	15.5%	18.2%	19.2%
3 levels 2/6/5	8.1%	11.7%	15.3%	18.1%	19.2%
3 levels 3/2/10	8.4%	12.0%	15.4%	18.2%	19.3%
4 levels 2/2/3/5	7.4%	11.2%	14.7%	18.0%	19.1%
4 levels 5/3/2/2	7.6%	11.2%	14.8%	18.0%	19.2%



Skewness	Theta 0.5	1	2	5	10
1 level 60	1.297	1.276	1.148	0.864	0.734
2 levels 2/30	1.573	1.475	1.249	0.978	0.839
2 levels 3/20	1.633	1.520	1.317	1.033	0.899
2 levels 6/10	1.697	1.526	1.330	1.052	0.904
2 levels 30/2	1.498	1.358	1.182	1.010	0.898
3 levels 2/2/15	1.571	1.746	1.428	1.026	0.819
3 levels 2/3/10	1.653	1.798	1.480	1.045	0.841
3 levels 2/6/5	1.640	1.781	1.543	1.062	0.844
3 levels 3/2/10	1.660	1.839	1.480	1.032	0.839
4 levels 2/2/3/5	1.724	1.965	1.591	1.096	0.884
4 levels 5/3/2/2	1.844	1.948	1.592	1.102	0.867



The above statistics confirm that:

- The structure of the aggregation tree e.g. the number of layers play a major role in the determination of the overall coefficient of variation and skewness and, as a consequence, of the Value at Risk 99.5%.
- The higher the number of layers is, the higher the skewness is but the lower the coefficient of variation. This is as expected in the first sections of this article.
- The coefficient of variation and the related diversification benefit play a bigger role than the skewness in the determination of the Value at Risk.

4. CONCLUSION

The aim of this article was to provide a few simple examples of skewness estimation and to see on two simple aggregation trees the impact of the change of the aggregation scheme on the skewness and hence, on the capital requirements. Overall, based on the conclusion and tests performed in section 3, we can see that the diversification benefit is playing a much more significant role than the skewness over the capital requirements. However, the examples in section 3 also suggest that the aggregation tree structure and, in particular, the number of layers of aggregation is a strong determinant of the Value at Risk as this structure influences significantly the coefficient of variation and the skewness of the overall risk distribution.

Even though such effects are expected, practitioners should still keep a cautious eye on the skewness impact over the capital requirements, in particular in extreme cases where diversification between risks is limited. In such case, the skewness may become more important than the diversification effect and the capital requirements may be strongly influenced by this effect.

In any case, even though the impact of the third and fourth moments in multivariate environment has started to be studied, it is still insufficiently reviewed and documented when copula calibration is done by practitioners. Therefore, further studies of such effects should be done.

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